

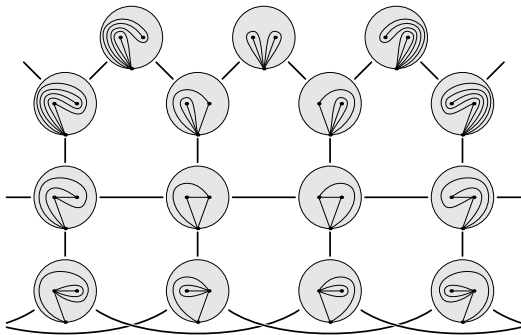
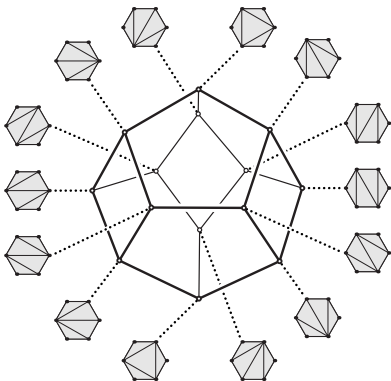
Flips and surface triangulations

Lionel Pournin
Université Paris 13

based on joint work with

Hugo Parlier
University of Luxembourg

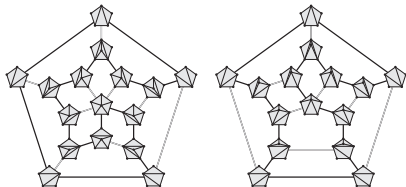
Zili Wang
Sun Yat-Sen University Shenzhen



Summary of the talk

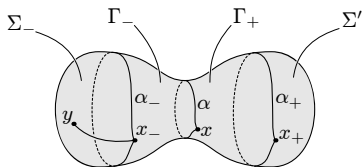
I. The geometric case

- I.1. The case of convex polygons
- I.2. General point configurations
 - a. Non-convex polygons
 - b. Interior points
 - c. Higher dimensions



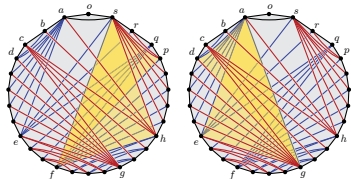
II. The topological case

- II.1. Flip-graphs of surfaces
- II.2. Diameters of modular flip-graphs
- II.3. Proof sketches for upper bounds



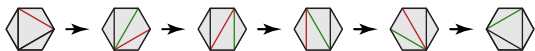
III. Path modeling as 3-dimensional triangulations

- III.1. Strong convexity
- III.2. Geodesicity implies flagness
- III.3. Consequences
 - a. Exact flip distance computation
 - b. Flip distance approximation
 - c. Strong convexity in the geometric case



I.1 Convex polygons

Remark. Using flips, one can measure a distance between two triangulations of a surface with a prescribed set of vertices.



Question (open). Given two triangulations of a convex n -gon, what is the distance between them in the flip-graph of the polygon?



Question (solved). What is the **largest possible** distance between any two triangulations in the flip-graph of a convex n -gon? In other words, what is the diameter (of the graph) of the associahedron?

Theorem (Sleator–Tarjan–Thurston 1988). The $(n - 3)$ -dimensional associahedron has diameter $2n - 10$ when n is large enough.

Theorem (P. 2014). *Large enough* means $n \geq 13$.

1.1 Convex polygons

Distances in the graph of the associahedron are also rotation distances between binary trees. Rotations are used in computer science to re-balance binary trees in order to improve data storage efficiency.

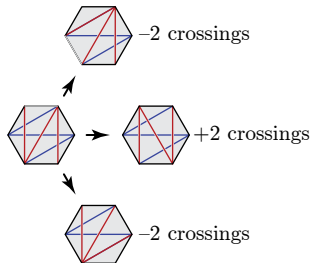
Distances in the graph of the associahedron are used to measure dissimilarity between two binary (phylogenetic) trees in computational biology. How can these distances be computed or, at least, approximated?

Heuristic that estimates $d(T_1, T_2)$

- (1) Flip an arc in T_1 such that the number of arc crossings with T_2 decreases,
- (2) Repeat until T_2 is reached.

Theorem (Hanke–Ottman–Schuierer 1997).

One can always flip some arc in T_1 such that the number of arc crossings with T_2 decreases after the flip.



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Unfortunately...

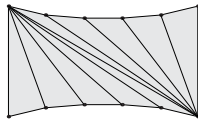
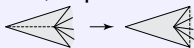
Theorem (Cleary–Maio 2018).

The distance estimation computed from this arc crossings-based heuristic is sometimes *one off* $d(T_1, T_2)$.

Can this get worse? Can the estimation be larger than $\alpha d(T_1, T_2)$ where $\alpha > 1$?

1.2. General point configurations

In the geometric setup, if interior points are allowed or the polygon is non-convex, flips are not always possible!



Remark. The diameter of the flip-graph of a double chain on n vertices is about n^2 whereas for a convex n -gon, this diameter is about $2n$.

Theorem (Lubiw–Pathak 2015/Pilz 2014). Flip-distance computation is NP -complete/ APX -hard when interior points are allowed.

Theorem (Aichholzer–Mulzer–Pilz 2015). Flip-distance computation is NP -complete for non-convex polygons.

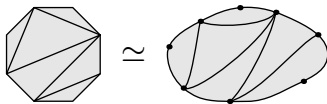
In dimension 5 and up, flip-graphs can be disconnected (Santos 2005).

Question. Are the flip-graph of the 3- and 4-dimensional point configurations always connected?

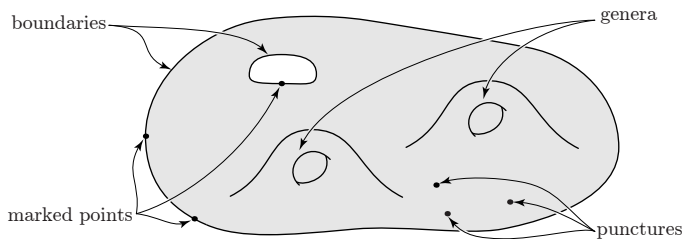
Theorem (P 2012). The flip-graph of the 4-dimensional cube is connected.

II.1. Flip-graphs of surfaces

Remark. The (geometric) case of a convex n -gon is *the same* as a topological disk with n marked points in its boundary.



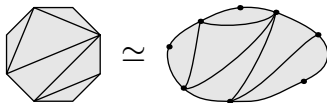
Question. Instead of a disk, can we pick a topological surface, possibly with punctures and at least one marked point in each boundary component?



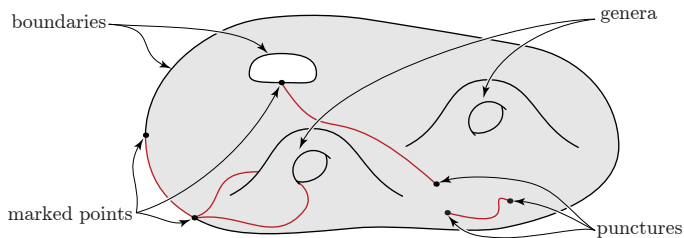
Lemma (Švarc 1955, Milnor 1968). The graph whose vertices are the triangulations of the surface Σ and whose edges correspond to flipping arcs is quasi-isometric to any Cayley graph of the mapping class group of Σ .

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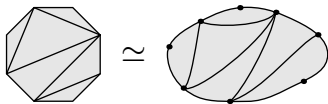
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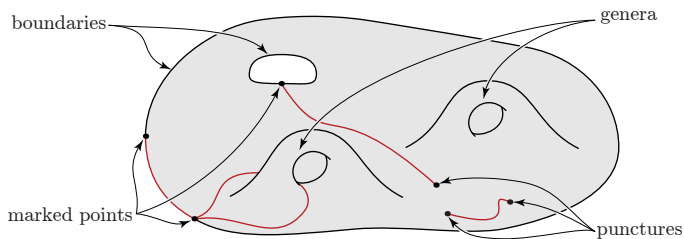
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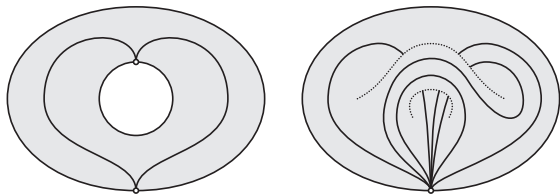


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Theorem (Disarlo–Parlier 2019). Given two triangulations T_1 and T_2 of the surface Σ , one can always flip some arc in T_1 such that the number of arc crossings with T_2 decreases after the flip.

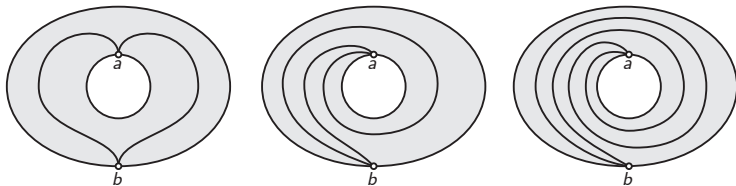
II.1. Flip-graphs of surfaces



Question. What about
(1) the connectedness
(2) the combinatorics
of the flip-graph of Σ ?

Theorem (Mosher 1988). The flip-graph of Σ is always connected.

...but it can be infinite (although the degree of its vertices is finite)...



Consider a surface Σ_n with n marked points on a privileged boundary.

Question. Call $\mathcal{MF}(\Sigma_n)$ the quotient of the flip-graph of Σ_n by its homeomorphisms. What is the diameter Δ of $\mathcal{MF}(\Sigma_n)$?

II.2. Diameters of modular flip-graphs

If the homeomorphisms can exchange the punctures:

Theorem (Frati 2017/Cardinal–Hoffmann–Kusters–Tóth–Wettstein 2018).

If Σ_n is a n -punctured sphere, then the diameter Δ of $\mathcal{MF}(\Sigma_n)$ satisfies

$$7n/3 + O(1) \leq \Delta \leq 5n - 23.$$

If the homeomorphisms cannot exchange the punctures:

- Disk: $\Delta = 2n - 10$ when $n > 12$ (P. 2014),
- Cylinder with a unique point on the inner boundary:

$$\Delta = \lfloor 5n/2 \rfloor - 2 \text{ (Parlier–P. 2017),}$$

- Surface with three boundaries:

$$\Delta = 3n + O(1) \text{ (Parlier–P. 2017),}$$

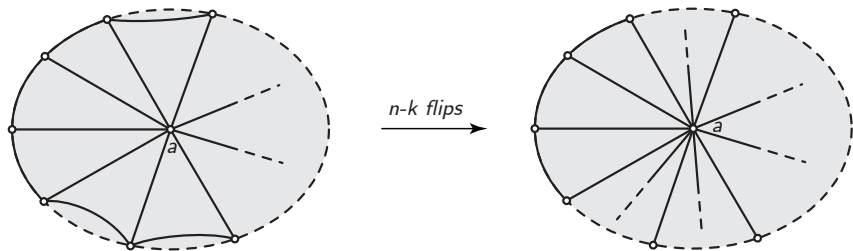
- Torus with a boundary containing all the vertices:

$$5n/2 + O(1) \leq \Delta \leq 23n/8 + O(1) \text{ (Parlier–P. 2018),}$$

- Punctured disk: $\Delta = 2n - 2$ (Parlier–P. 2018).

II.2. Diameters of modular flip-graphs

Theorem (Parlier–P. 2018). The maximal distance between two triangulations of the punctured disk is exactly $2n - 2$.

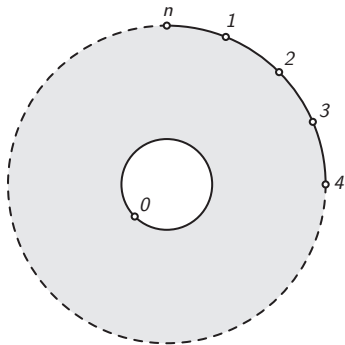


Remarks.

- There are n (non-boundary) arcs in such a triangulation.
- The puncture cannot disappear and is incident to at least one edge.
- The puncture can be incident to exactly one edge of the triangulation.

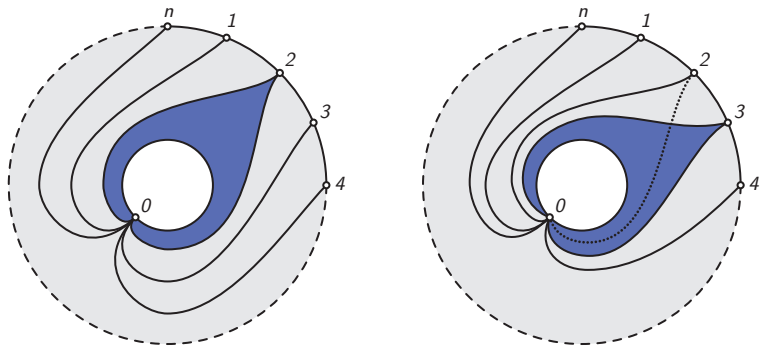
II.2. Diameters of modular flip-graphs

Consider a cylinder with n vertices in the outer boundary labeled 1 to n clockwise and one vertex labeled 0 in the inner boundary.



II.2. Diameters of modular flip-graphs

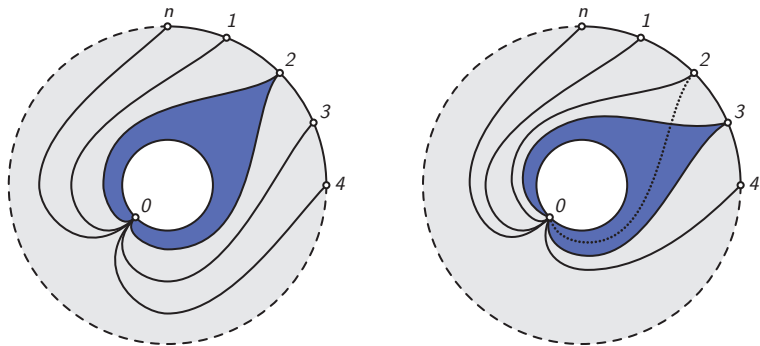
There is a (blue) triangle in any triangulation, whose vertices are 0 and a boundary vertex x . A triangulation has exactly $n + 1$ interior edges.



Hence at most $n - 1$ flips are required to insert all the edges incident to 0.

II.2. Diameters of modular flip-graphs

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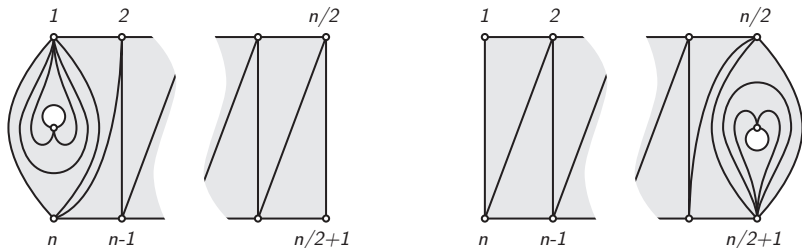


Boundary vertices are separated by at most $n/2 - 1$ vertices. Bringing the blue triangles together thus requires at most $n/2$ flips.

II.2. Diameters of modular flip-graphs

Theorem (Parlier–P. 2017). The distance between two triangulations of the cylinder is never greater than $\lfloor 5n/2 \rfloor - 2$

This upper bound is sharp for all $n \geq 1$.



These two triangulations have flip distance exactly $\lfloor 5n/2 \rfloor - 2$.

Remark. One obtains two triangulations of the punctured disk with flip distance $2n - 2$ by shrinking the inner boundary to a puncture and identifying the two arcs incident to that puncture.

II.2. Diameters of modular flip-graphs

Remarks.

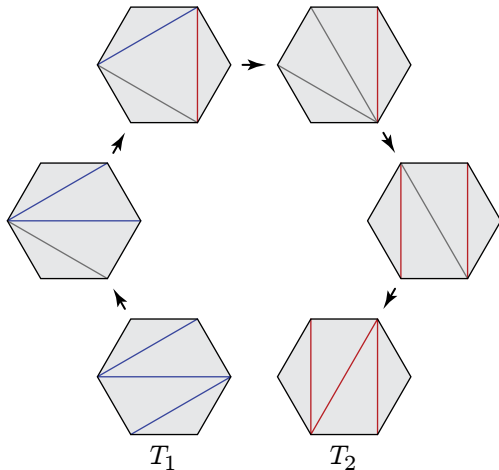
- We have observed growth rates of
2 (disk, punctured disk),
 $5/2$ (cylinder), and
3 (surface with 3 boundaries).
- Growth rates cannot exceed 4 (Parlier–P. 2017)...
- ...or be between 2 and $5/2$ (Parlier–P. 2018).

Open questions.

- What are all the possible growth rates?
- What is the *exact* growth rate for the bordered torus?
- What diameters can we compute when we allow the topology of the surface to vary (and not only n)?
- ...and when vertices are not marked?

III. A topological model for paths

Remark. A path between triangulations T_1 and T_2 in the graph of the associahedron can be thought of as a certain type of 3-dimensional triangulation.



Remark. A *blow-up triangulation* is not a usual triangulation:

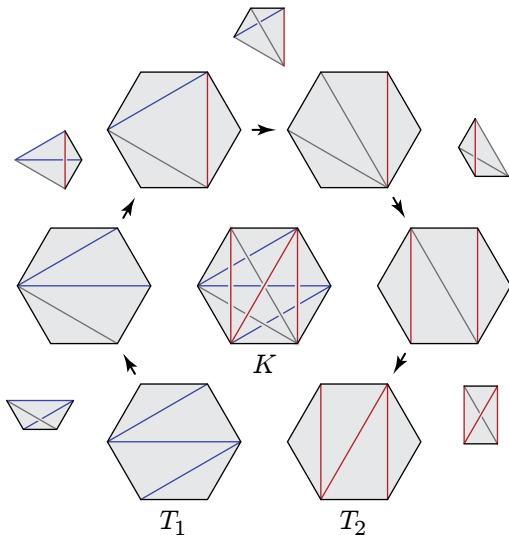
- (1) Multiple arcs are allowed,
- (2) Two tetrahedra can be glued along the union of two triangles.

Remark. $d(T_1, T_2)$ is also the number of tetrahedra required to fill a triangulated sphere S .

...and an upper bound on the L^1 norm of the 3-chains whose boundary is a 2-cycle corresponding to the triangulation of S .

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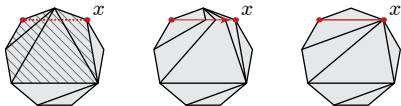
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III.1. Strong convexity

Lemma (Sleator–Tarjan–Thurston 1988). The triangulations that contain a given arc ε induce a strongly convex subgraph in the graph of the associahedron: ε is not removed along any geodesic between two such triangulations.

Proof. Consider a path between two triangulations that contain ε and project each triangulation T in that path as follows.



Consecutive triangulations in the path are projected to either

- (1) two triangulations related by a flip or
- (2) the same triangulation.

A flip that removes ε is of the second kind. If there is such a flip in the considered path, the projected path is shorter. \square

Theorem (Disarlo–Parlier 2019). The same holds for topological surfaces.

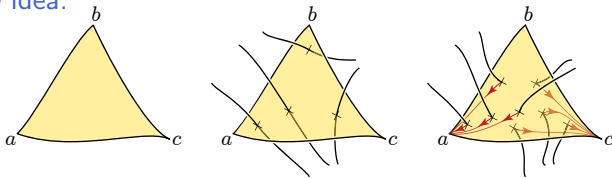
III.2. Geodesicity implies flagness

Theorem (P.-Wang 2021). The blow-up triangulations that correspond to a geodesic path in the graph of the associahedron are *flag*.

A blow-up triangulation K is flag when three arcs in K that form a cycle always bound a triangle of K .

In other words: if the three edges of a triangle abc appear in possibly distinct triangulations along a geodesic path in the graph of the associahedron, then abc itself appears along that path.

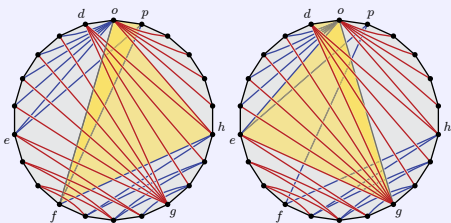
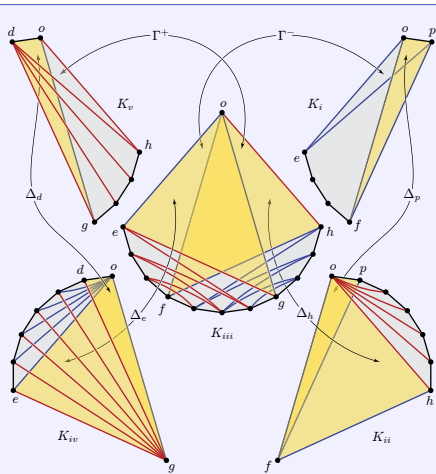
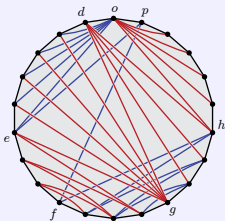
Proof rough idea.



Theorem (P.-Wang 2021). Blow-up triangulations corresponding to geodesic paths remain flag when the initial convex polygon (or topological disk) is replaced by a convex polygon with well-placed flat vertices.

III.3. Some consequences

Theorem (P.-Wang 2021). The arc crossings based distance estimate method sometimes overestimates distances by a factor that can get arbitrarily close to $3/2$ both in the cases of associahedra and topological surfaces.



III.3. Some consequences

Theorem (P.-Wang 2021). The subgraph induced by the triangulations that contain a given arc is not always strongly convex in the limit case of a **geometric** convex polygon with as few as two flat vertices or punctures.

