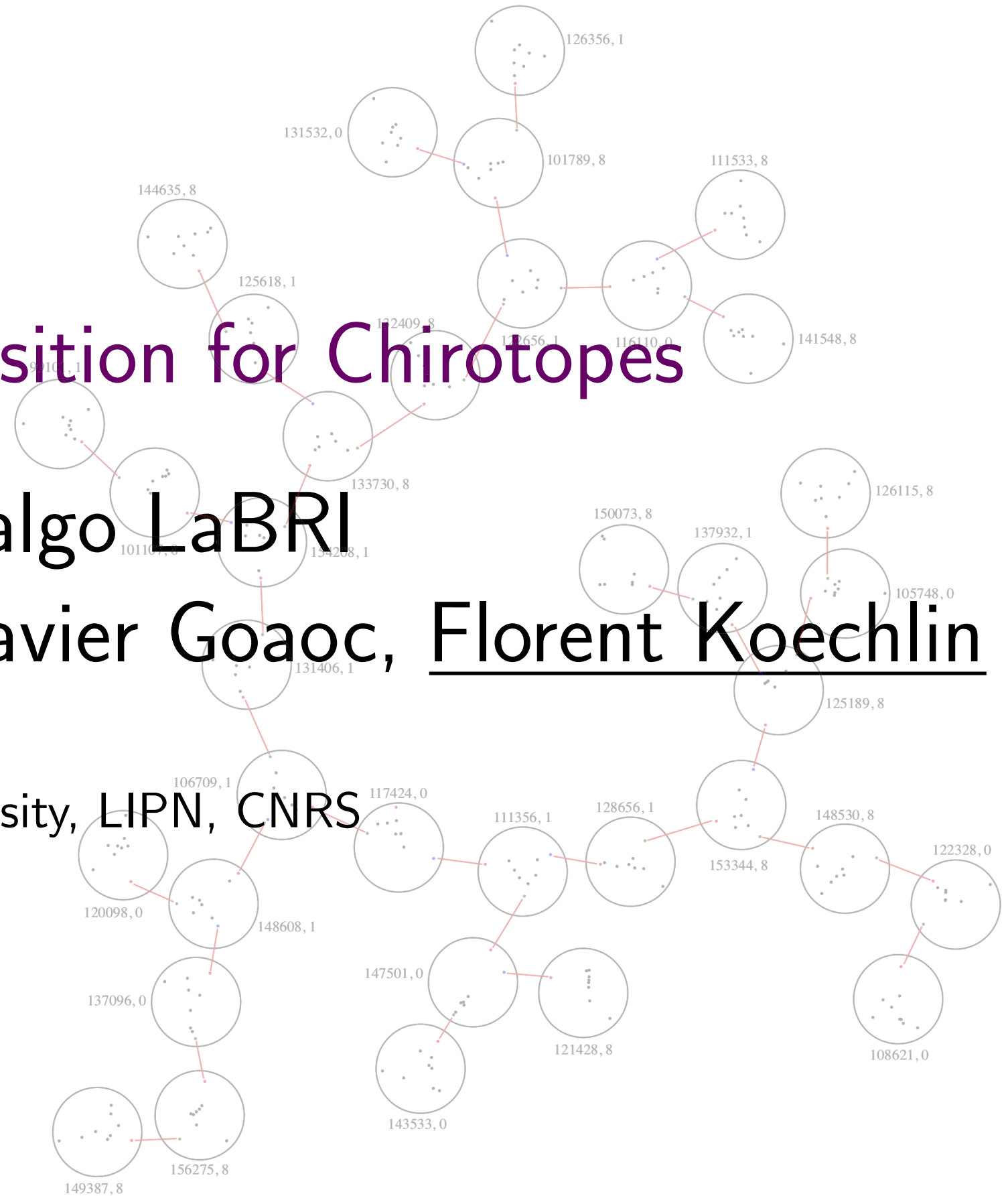
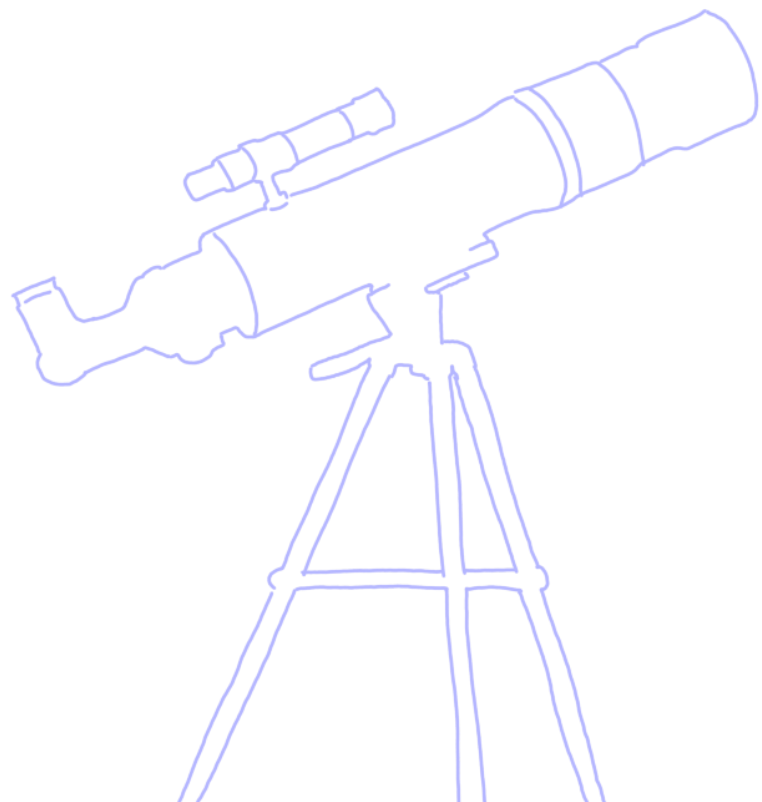


A Canonical Tree Decomposition for Chirotopes

Séminaire Combalgo LaBRI

Mathilde Bouvel, Valentin Féray, Xavier Goaoc, Florent Koechlin

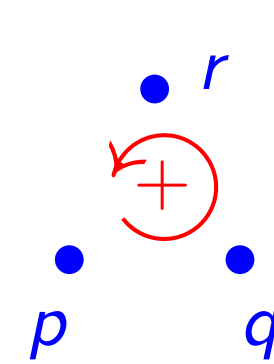
Sorbonne Paris Nord University, LIPN, CNRS

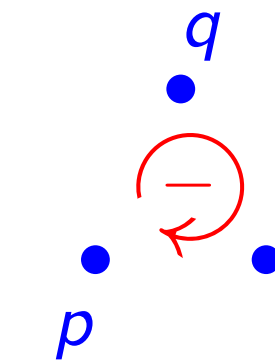


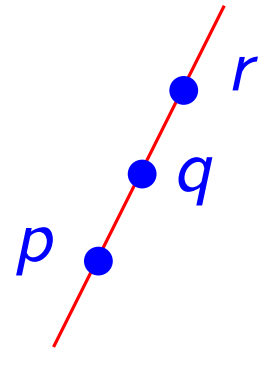
Chirotope

Let \mathcal{P} be a set of labeled points

$$\chi_{\mathcal{P}}(p, q, r) = \text{sign} \begin{vmatrix} x_p & x_q & x_r \\ y_p & y_q & y_r \\ 1 & 1 & 1 \end{vmatrix}$$


$$\chi_{\mathcal{P}}(p, q, r) = +1$$

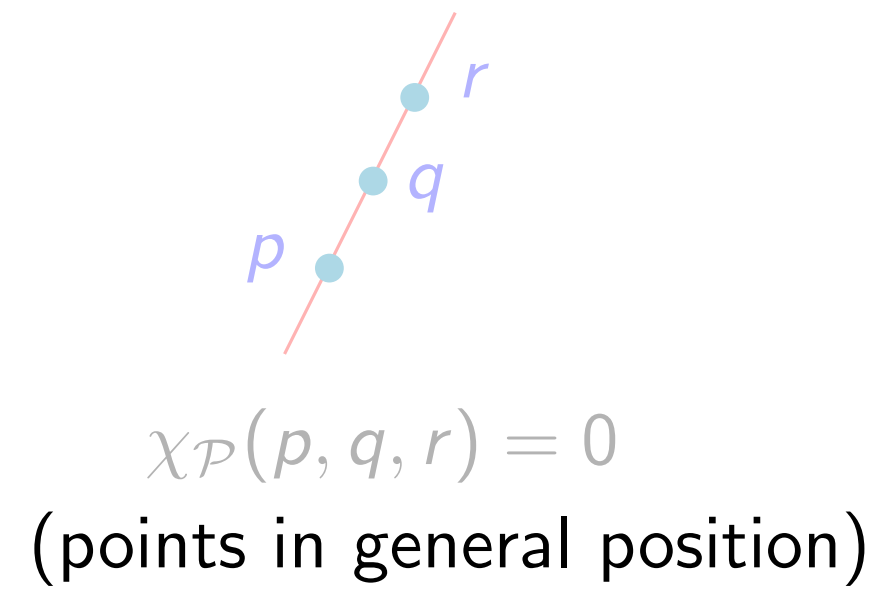
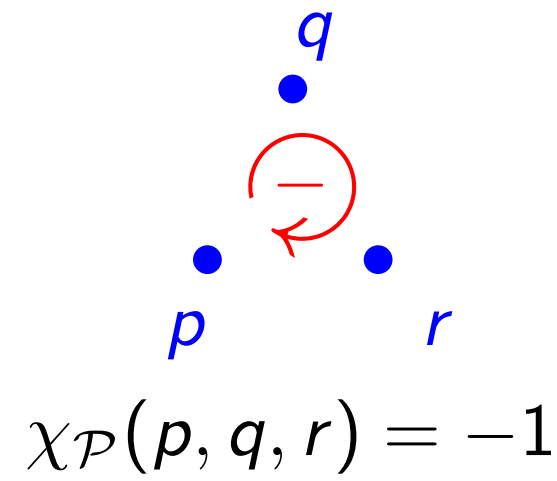
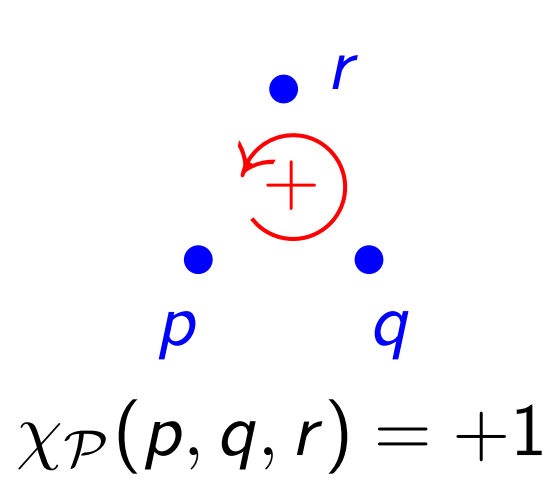

$$\chi_{\mathcal{P}}(p, q, r) = -1$$


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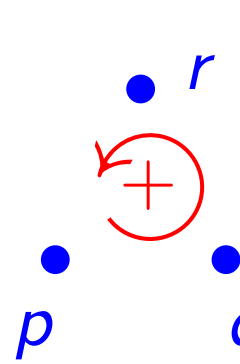
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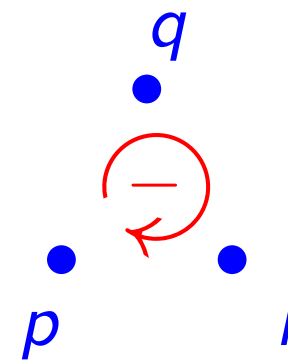
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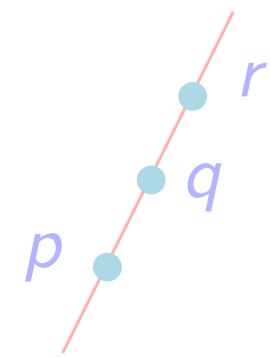
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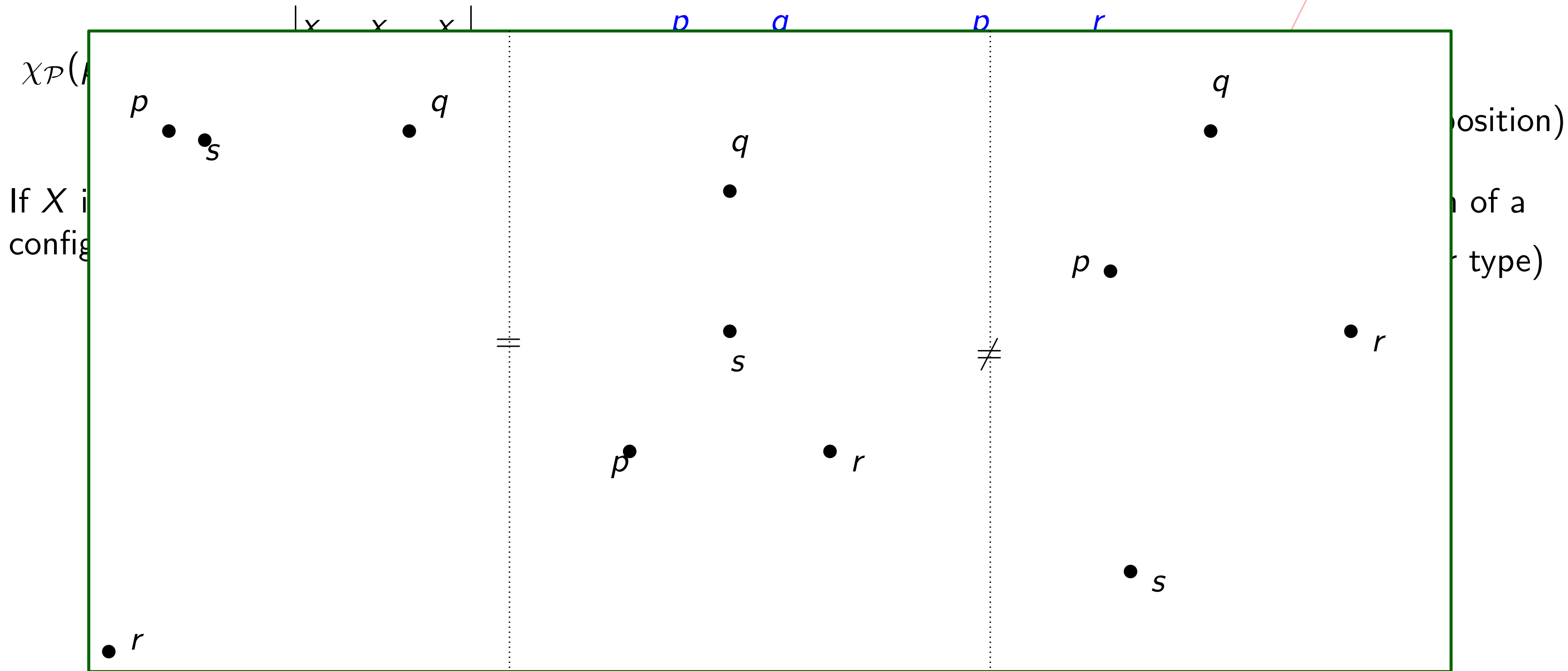
If X is a set of labels, $f : (X)_3 \rightarrow \{-1, 1\}$ is a **realizable chirotope** if it is the orientation function of a configuration of points labeled by X .

(chirotope = labeled order type)

Realizable chirotopes = combinatorial representation of configuration of points

Chirotope

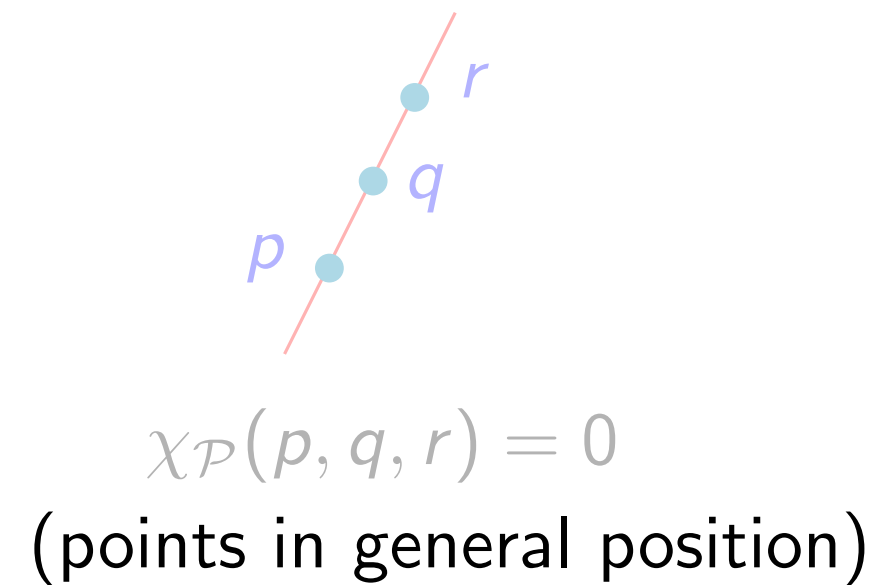
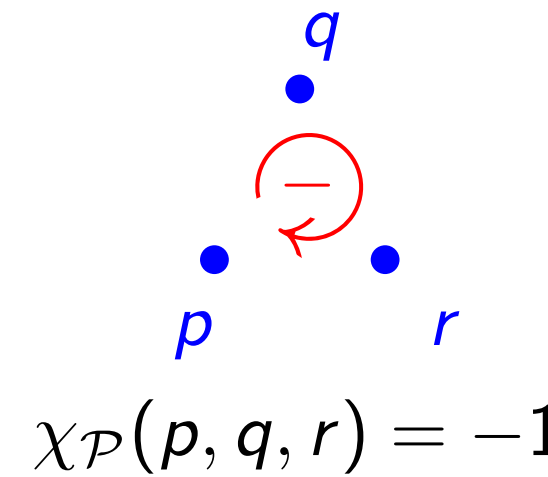
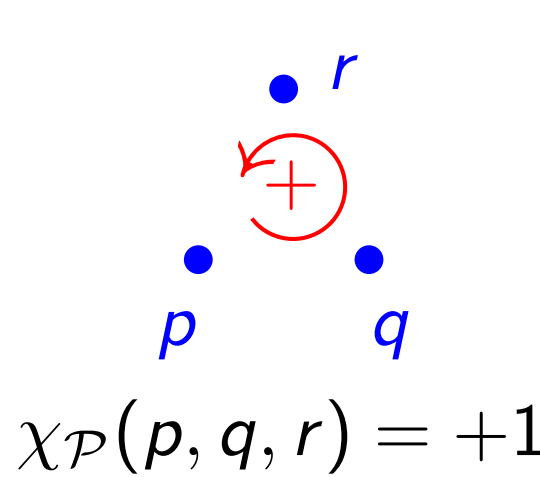
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Chirotope

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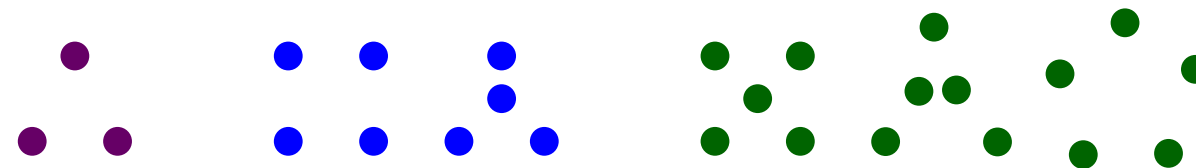
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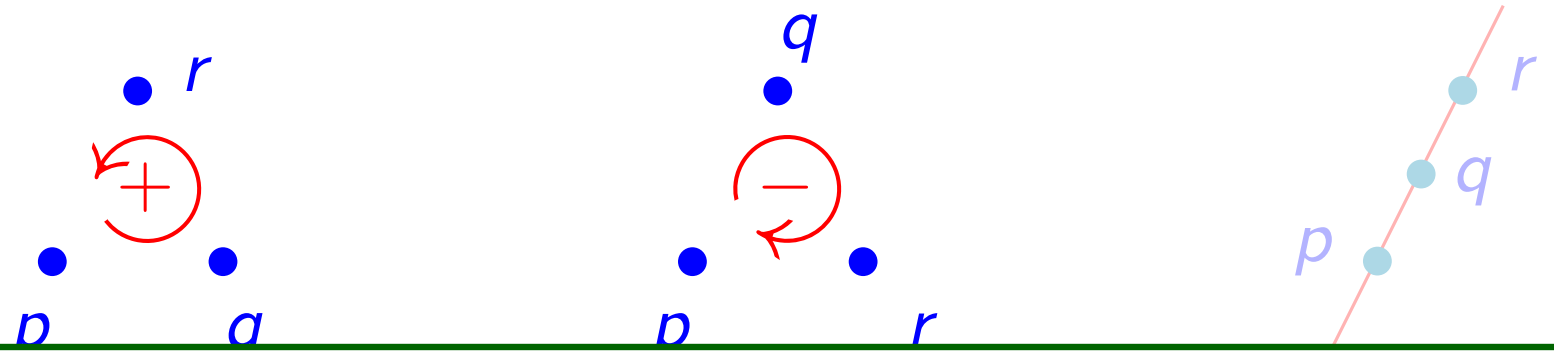
Reduces the diversity of points in \mathbb{R}^2 to a **finite** number of configurations...



... while preserving many **geometric properties**, like convex hull, crossing properties, number of triangulations, ...

Chirotope

Let \mathcal{P} be a set of labeled points



$\chi_{\mathcal{P}}(A$

If X is
config

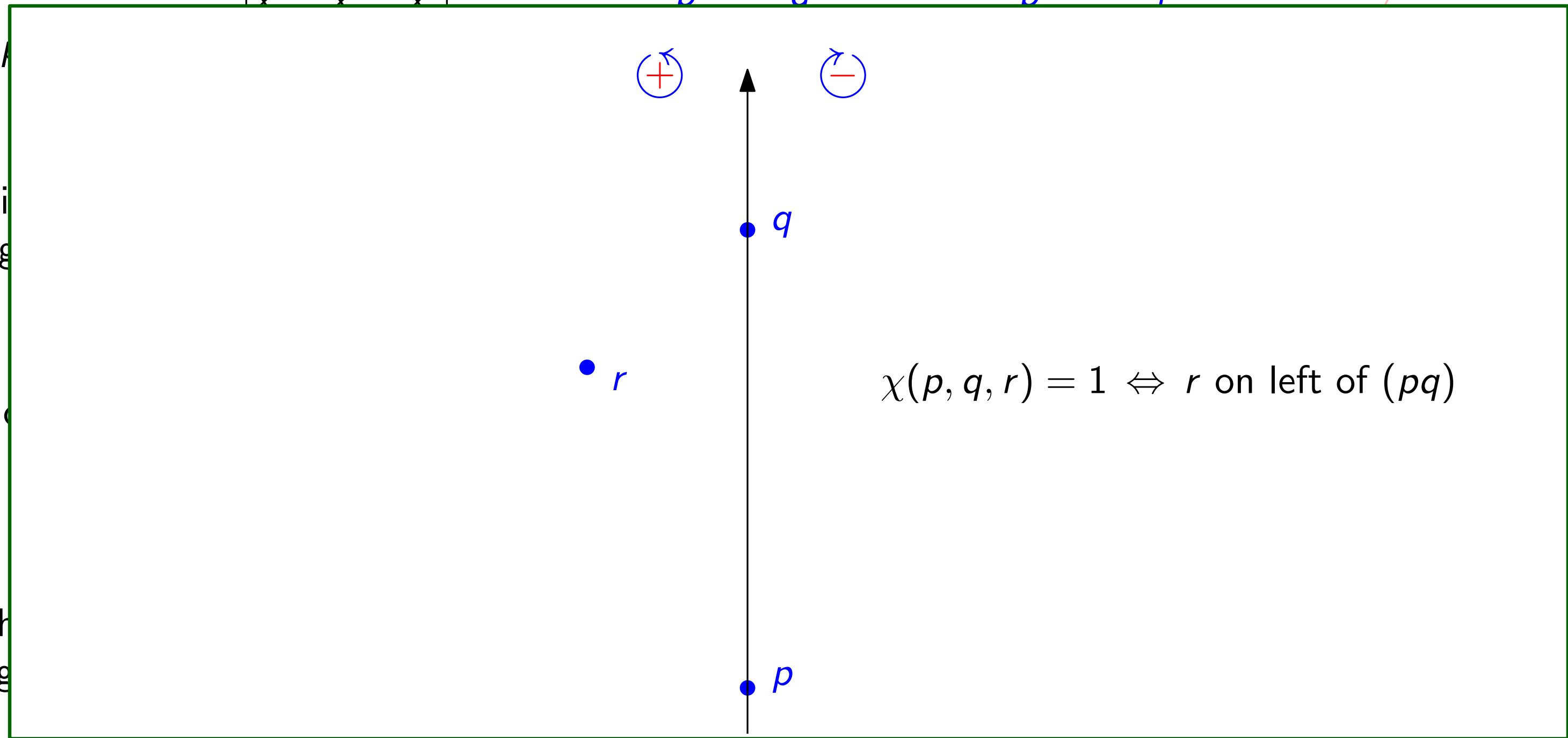
Reduc

... wh
triang

osition)

of a

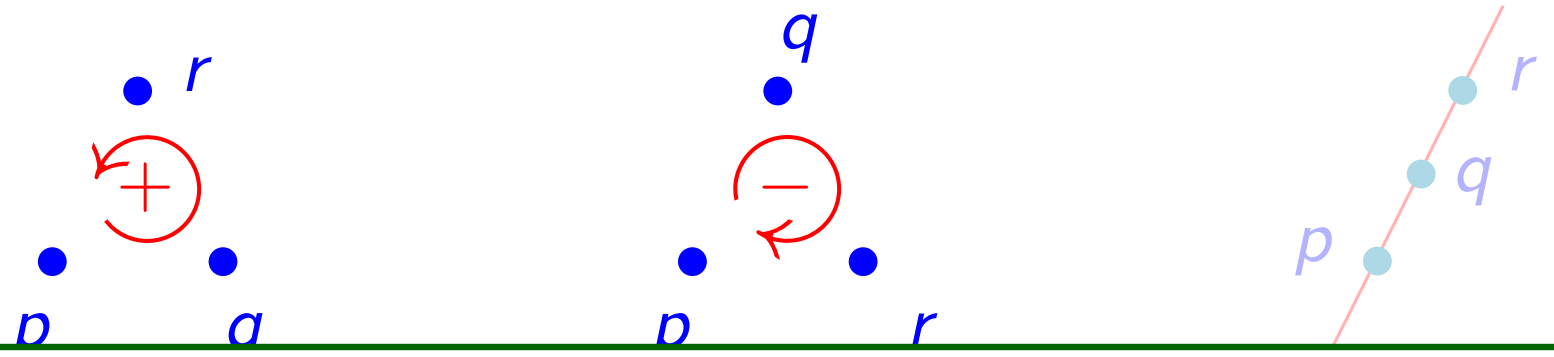
r type)



$$\chi(p, q, r) = 1 \Leftrightarrow r \text{ on left of } (pq)$$

Chirotope

Let \mathcal{P} be a set of labeled points



$|x \quad x \quad x|$

$\chi_{\mathcal{P}}(A$ (position)

If X is a configuration of a (type)

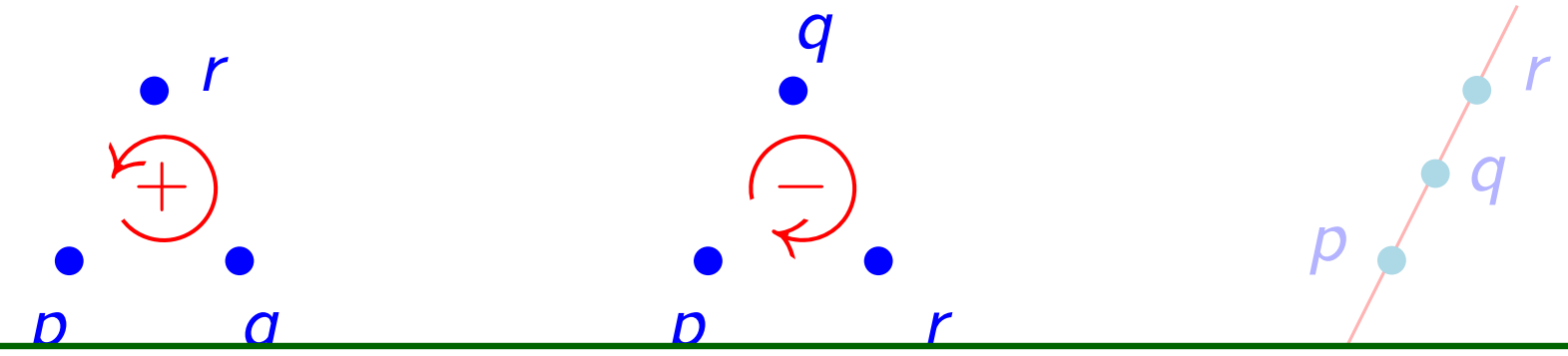
Reduced

... which triangulation

$[p, q] \in \text{Conv}(\mathcal{P}) \Leftrightarrow \forall x \in X \setminus \{p, q\}, \chi(p, q, x) = cst$

Chirotope

Let \mathcal{P} be a set of labeled points

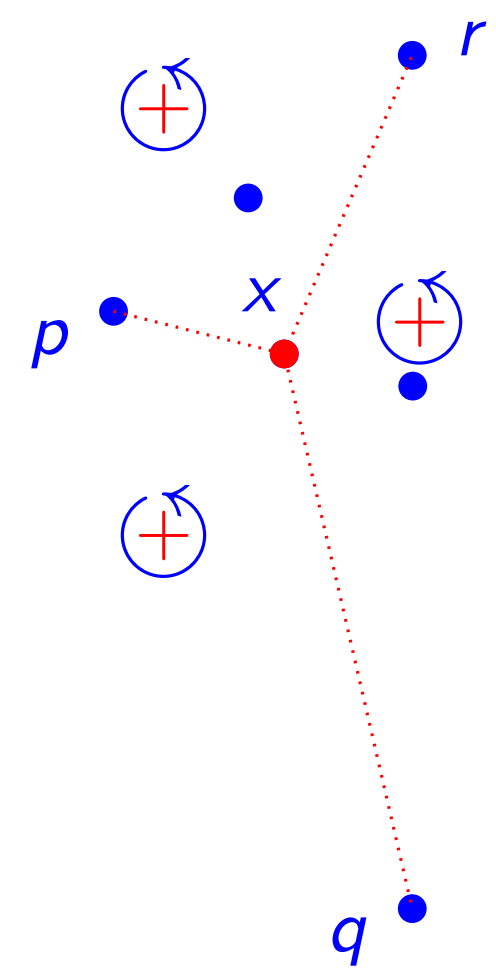


$\chi_{\mathcal{P}}(p, q, r)$

If X is a configuration

Reduced

... which triangulates

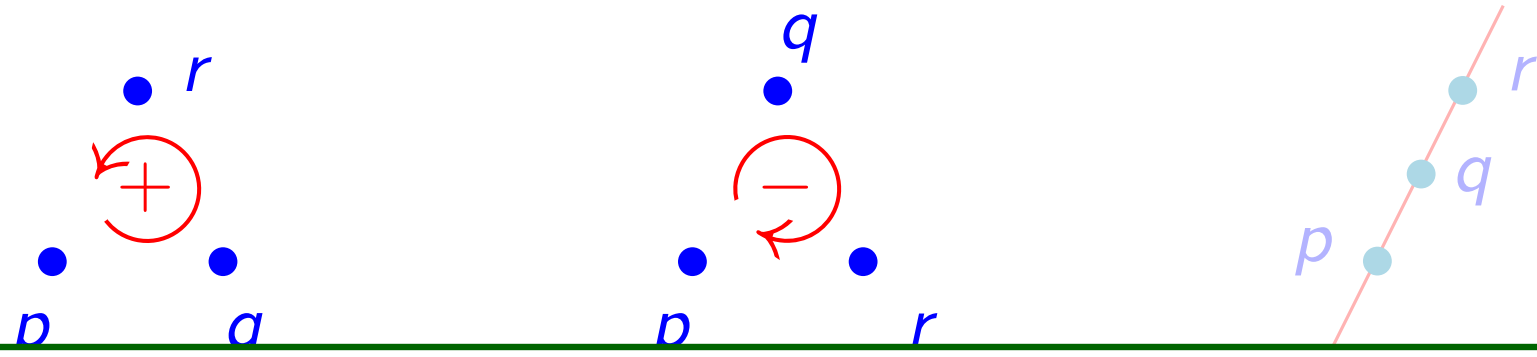


$$x \in \Delta(p, q, r) \Leftrightarrow \chi_{\mathcal{P}}(p, q, x) = \chi_{\mathcal{P}}(q, r, x) = \chi_{\mathcal{P}}(r, p, x)$$

(orientation)
(of a
type)

Chirotope

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$\chi_{\mathcal{P}}(A$

If X is a
config

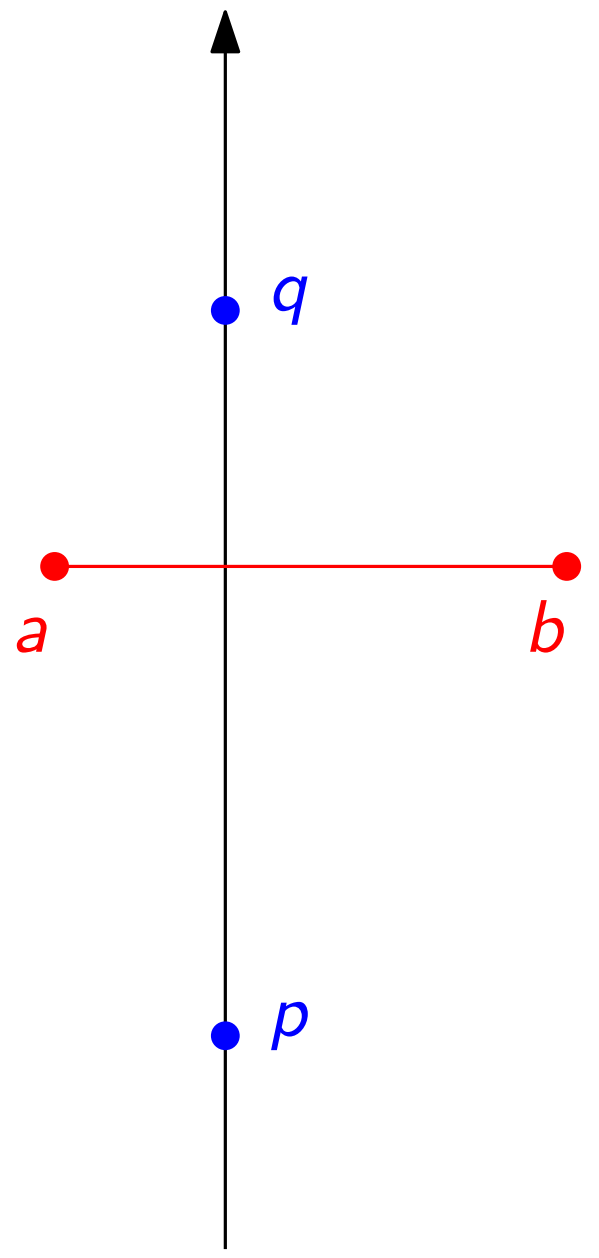
Reduc

... wh
triang

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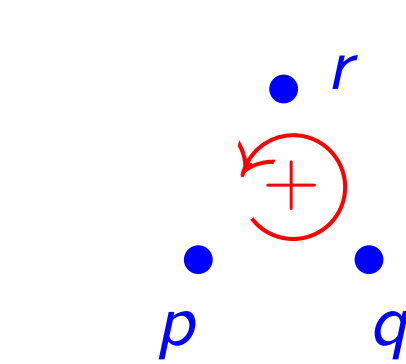
$$(p, q) \text{ separates } [a, b] \Leftrightarrow \chi(p, q, a) = -\chi(p, q, b)$$

$$[p, q] \text{ and } [a, b] \text{ intersect } \Leftrightarrow \dots$$

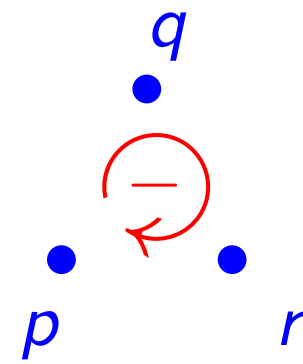
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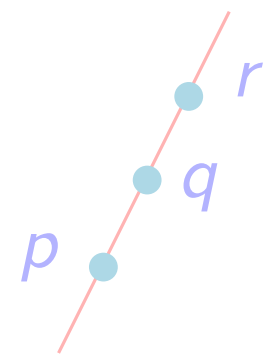
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Reduces the diversity of points in \mathbb{R}^2 to a **finite** number of configurations...



... while preserving many **geometric properties**, like convex hull, crossing properties, number of triangulations, ...

Possible **applications**: entries of algorithms using only the orientation, build a database of every relevant configurations of points (testing conjectures, benchmarks, ...)

Intractability of realizable chirotopes

Enumeration is hard. Number and list* of unlabeled chirotopes is known up to size 11 :

1, 2, 3, 16, 135, 3 315, 158 817, 14 309 547, 2 334 512 907 [Aichholzer et al 2002]

Asymptotics is hard. Number t_n of labeled chirotopes of size n behaves like $t_n = n^{4n + \Theta(n/\log n)}$

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Random exploration is hard

realizability is NP-hard [Shor 91], and is in fact ETR-complete [Mnëv 86]

drawing random points in the square leads a bad distributions [Goaoc and Welzl 23]

exhaustive polynomial time random generator \Leftrightarrow Realizability is in NP [Discussion with Emo Welzl]

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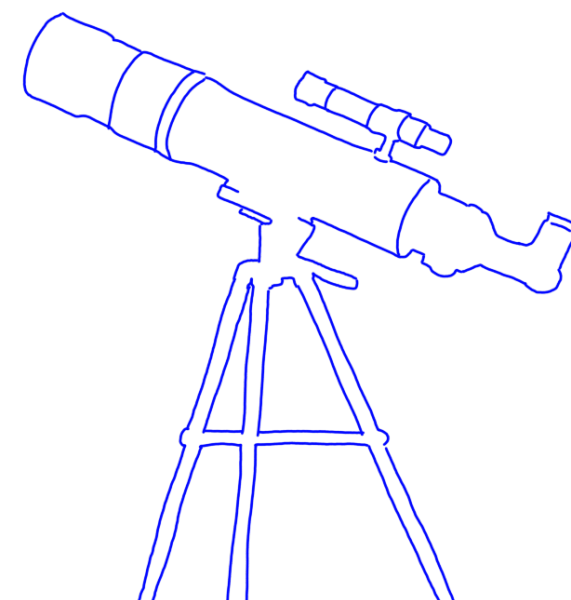
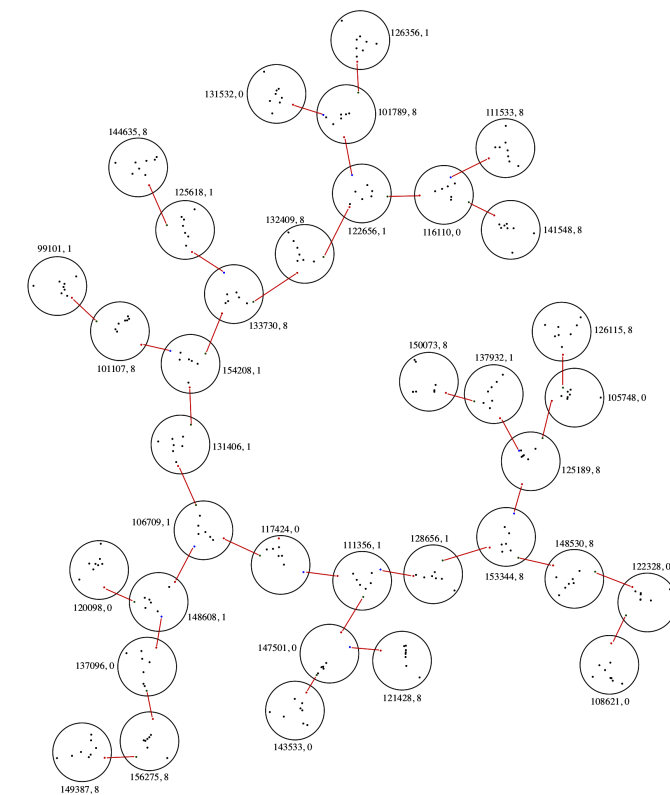
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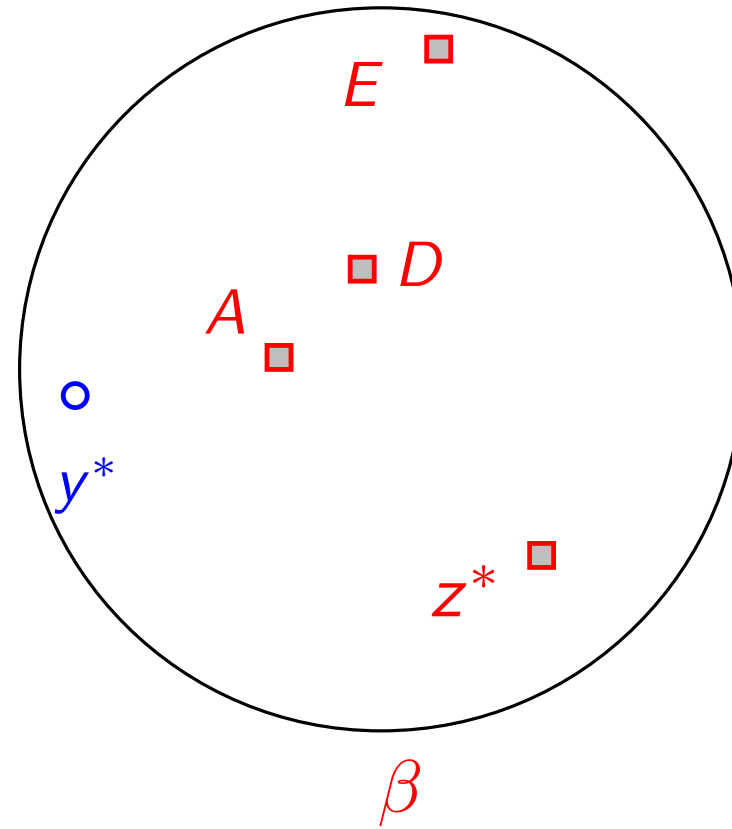
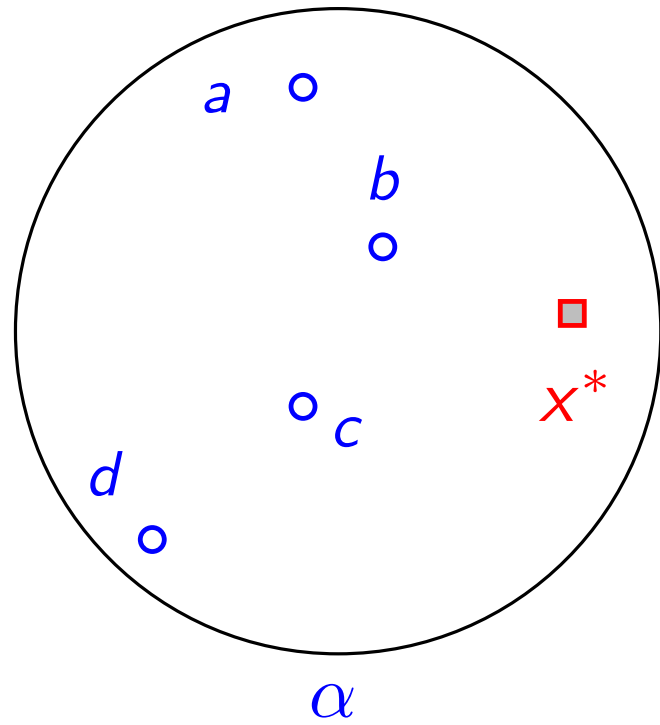
Conclusion big universe, we probably can only build telescopes

In this talk, we define an operation that builds big chirotopes by **assembling** smaller chirotopes together



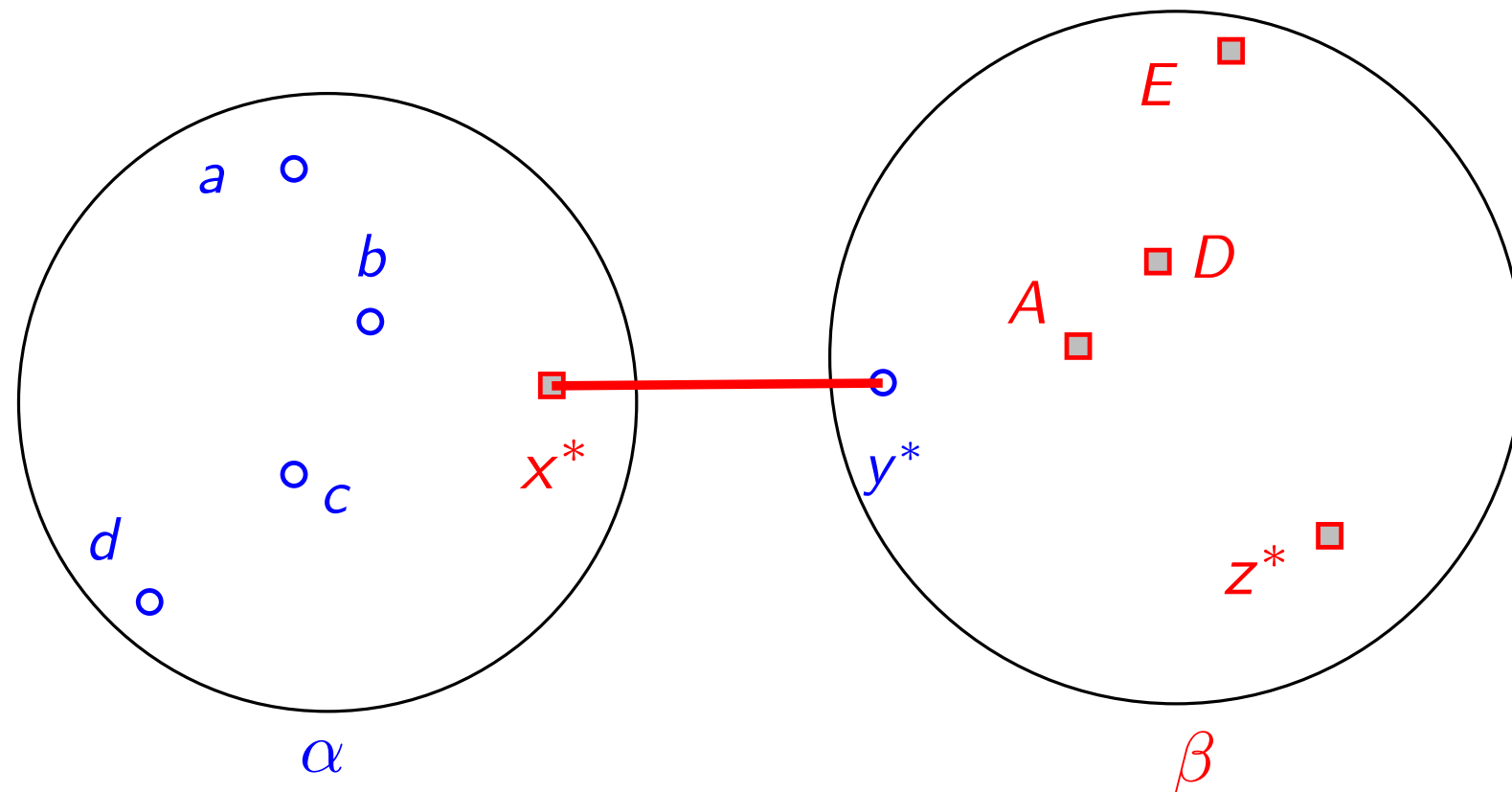
Bowtie construction

α realizable chirotope on $X \cup \{x^*\}$, and β on $Y \cup \{y^*\}$



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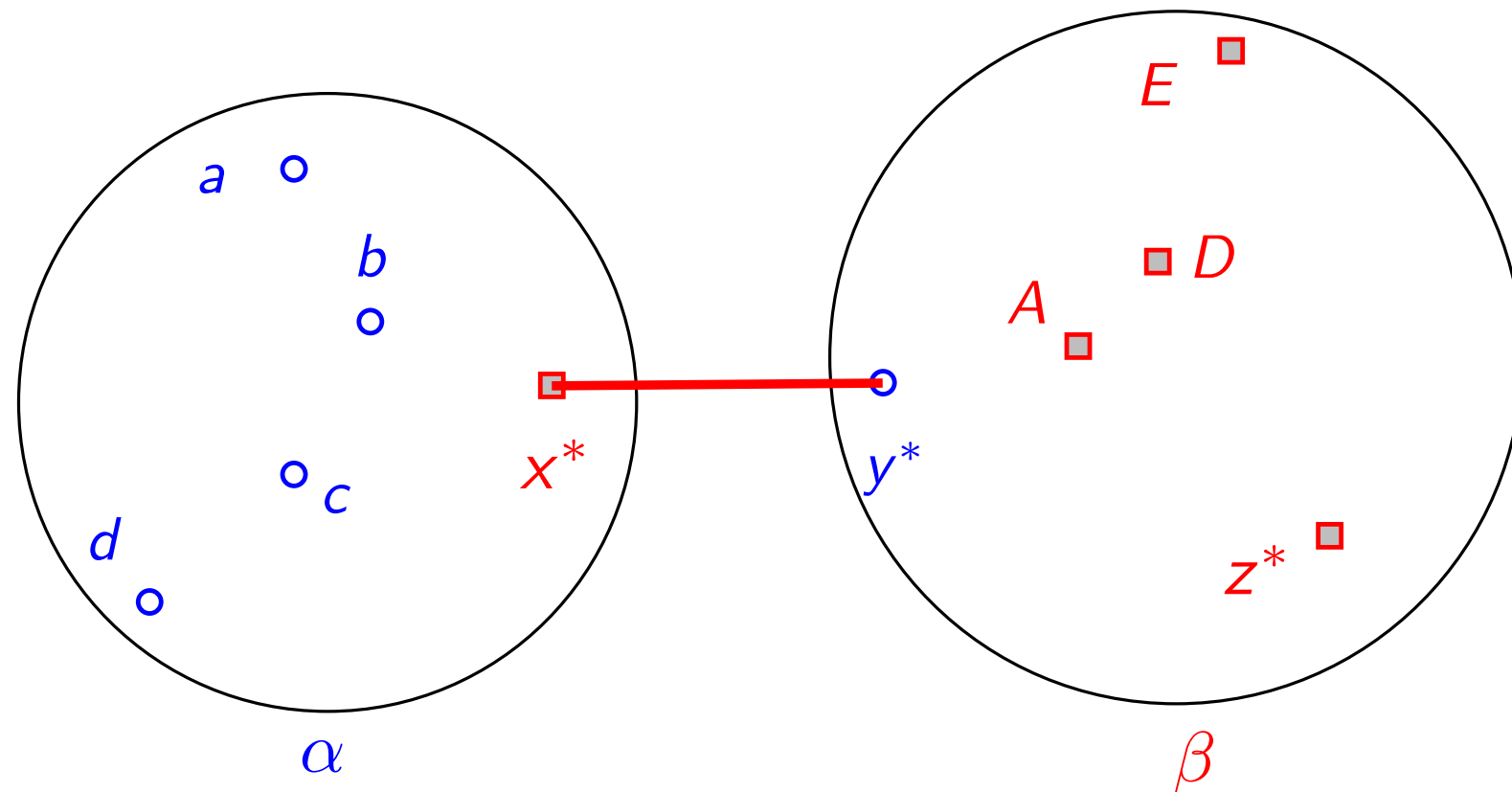
$$\begin{aligned} \alpha(a, d, x^*) &= \kappa(a, d, E) \\ &= \kappa(a, d, A) \\ &= \kappa(a, d, D) \\ &= \kappa(a, d, z^*) \end{aligned}$$

the **bowtie** $\kappa \stackrel{\text{def}}{=} \alpha_{x^*} \bowtie_{y^*} \beta$ is defined on $X \cup Y$ by:

{	$\kappa(x_1, x_2, x_3)$	$=$	$\alpha(x_1, x_2, x_3)$	if x_1, x_2, x_3 are all in X ;
	$\kappa(x_1, x_2, y)$	$=$	$\alpha(x_1, x_2, x^*)$	if x_1, x_2 are in X and y is in Y ;
	$\kappa(x, y_2, y_3)$	$=$	$\beta(y^*, y_2, y_3)$	if x is in X and y_2, y_3 are in Y ;
	$\kappa(y_1, y_2, y_3)$	$=$	$\beta(y_1, y_2, y_3)$	if y_1, y_2, y_3 are all in Y .

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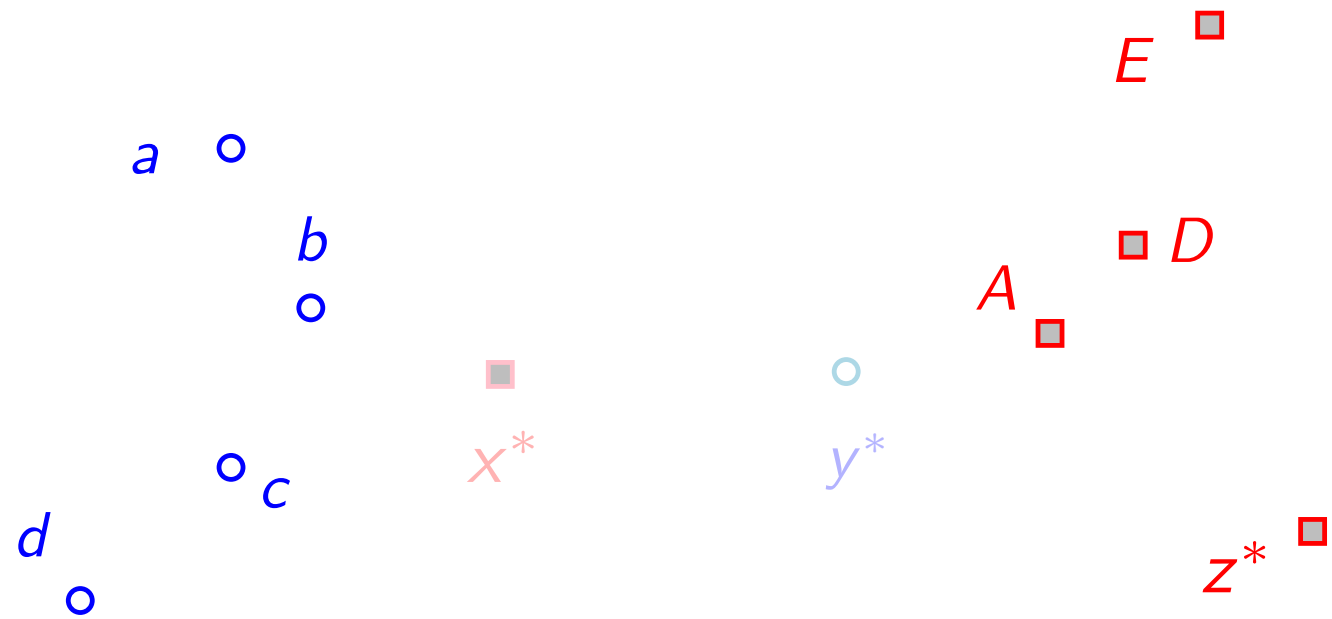
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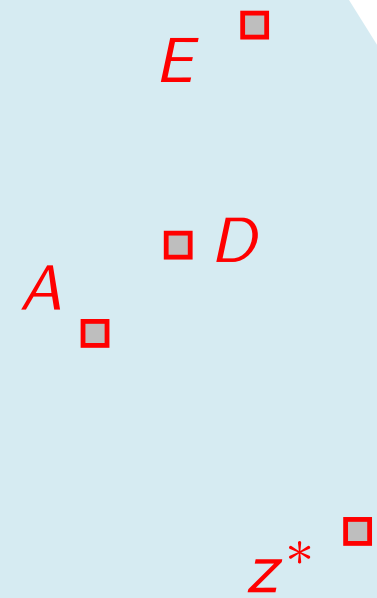
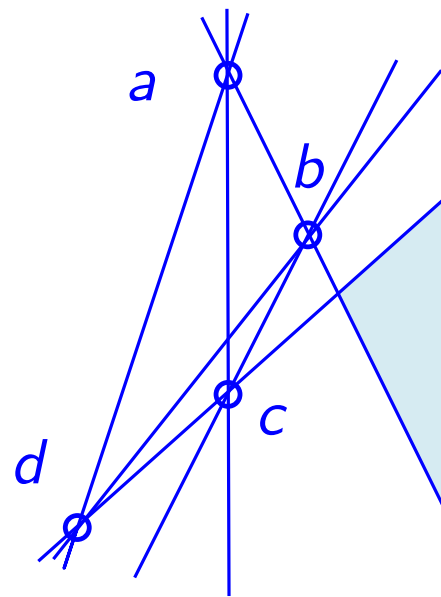
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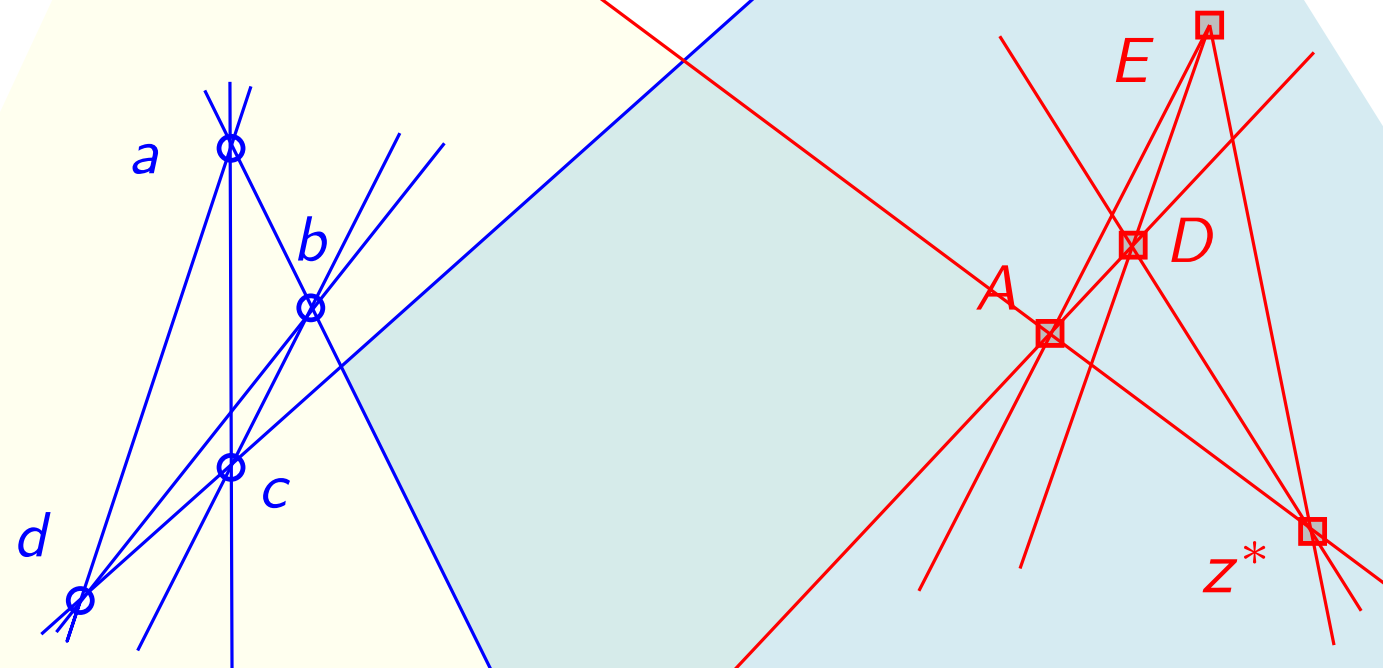
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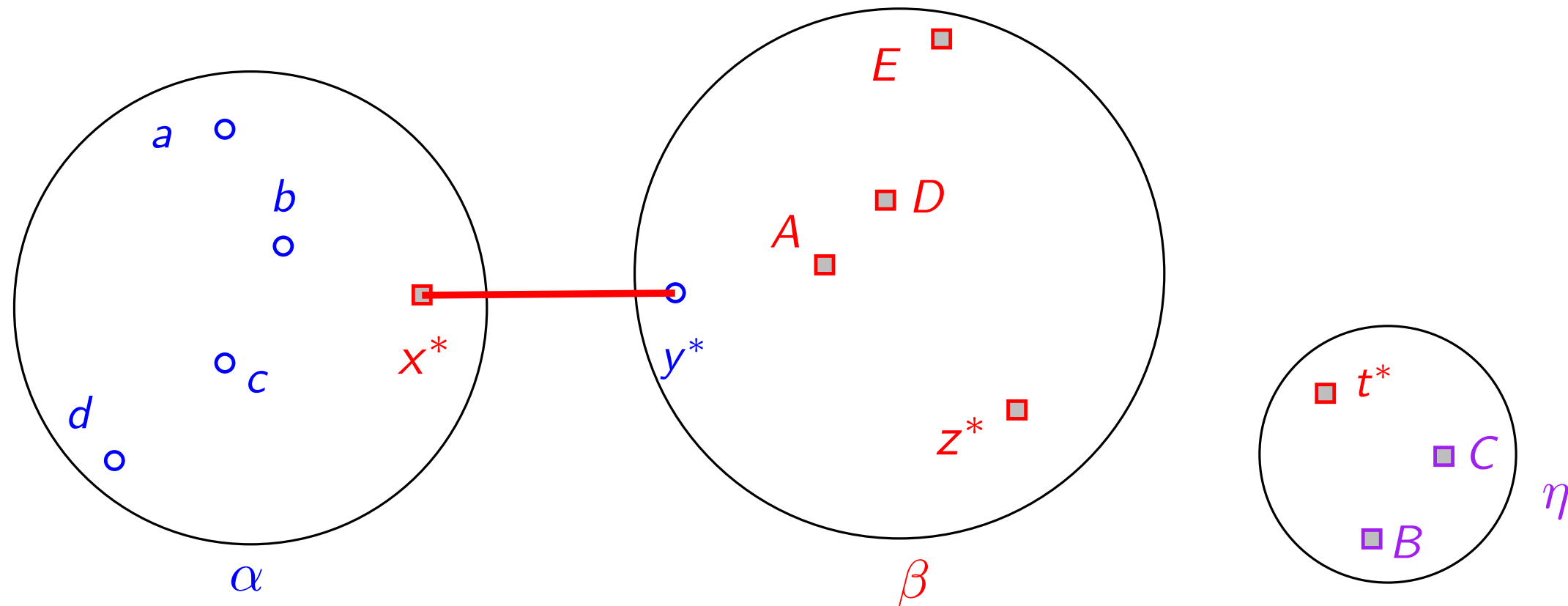
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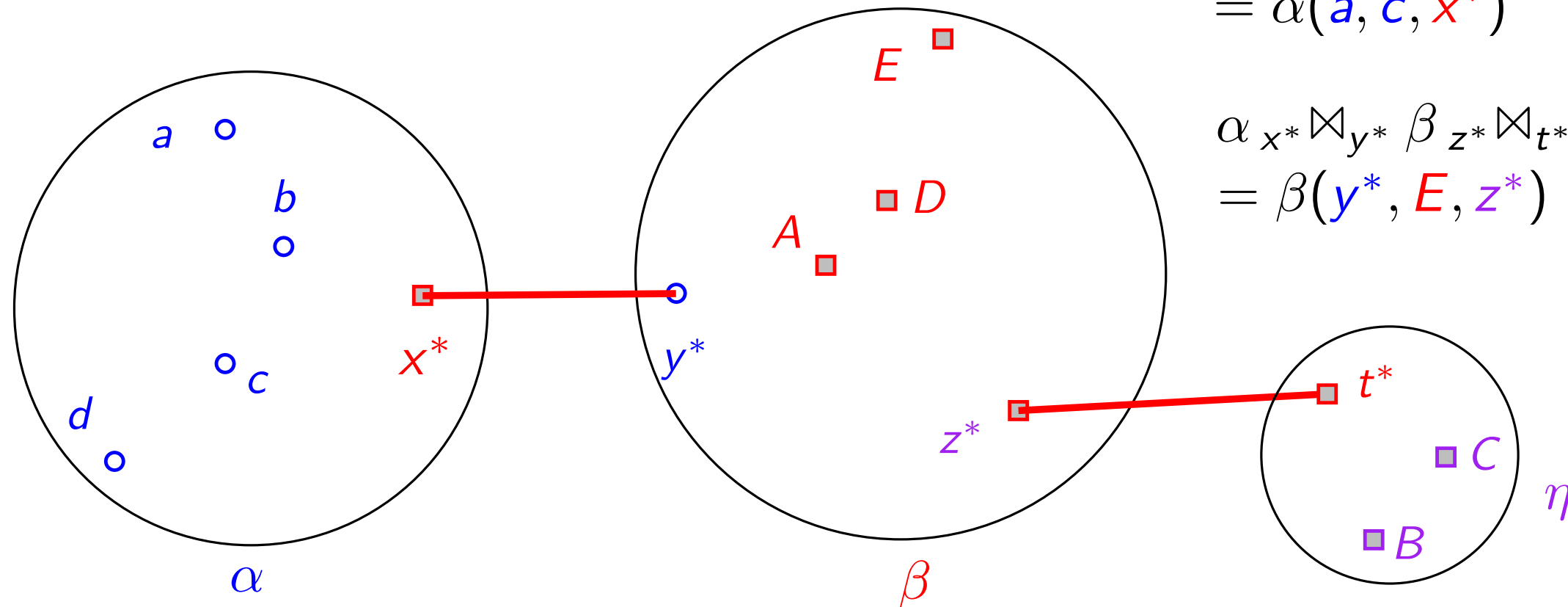
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\bowtie is associative: we can iterate the process to build a whole chirotope tree

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$$\alpha_{x^*} \bowtie_{y^*} \beta_{z^*} \bowtie_{t^*} \eta(a, c, B) = \alpha(a, c, x^*)$$

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the **bowtie** $\kappa \stackrel{\text{def}}{=} \alpha_{x^*} \bowtie_{y^*} \beta$ is defined on $X \cup Y$ by:

$\kappa(x_1, x_2, x_3)$	$=$	$\alpha(x_1, x_2, x_3)$	if x_1, x_2, x_3 are all in X ;
$\kappa(x_1, x_2, y)$	$=$	$\alpha(x_1, x_2, x^*)$	if x_1, x_2 are in X and y is in Y ;
$\kappa(x, y_2, y_3)$	$=$	$\beta(y^*, y_2, y_3)$	if x is in X and y_2, y_3 are in Y ;
$\kappa(y_1, y_2, y_3)$	$=$	$\beta(y_1, y_2, y_3)$	if y_1, y_2, y_3 are all in Y .

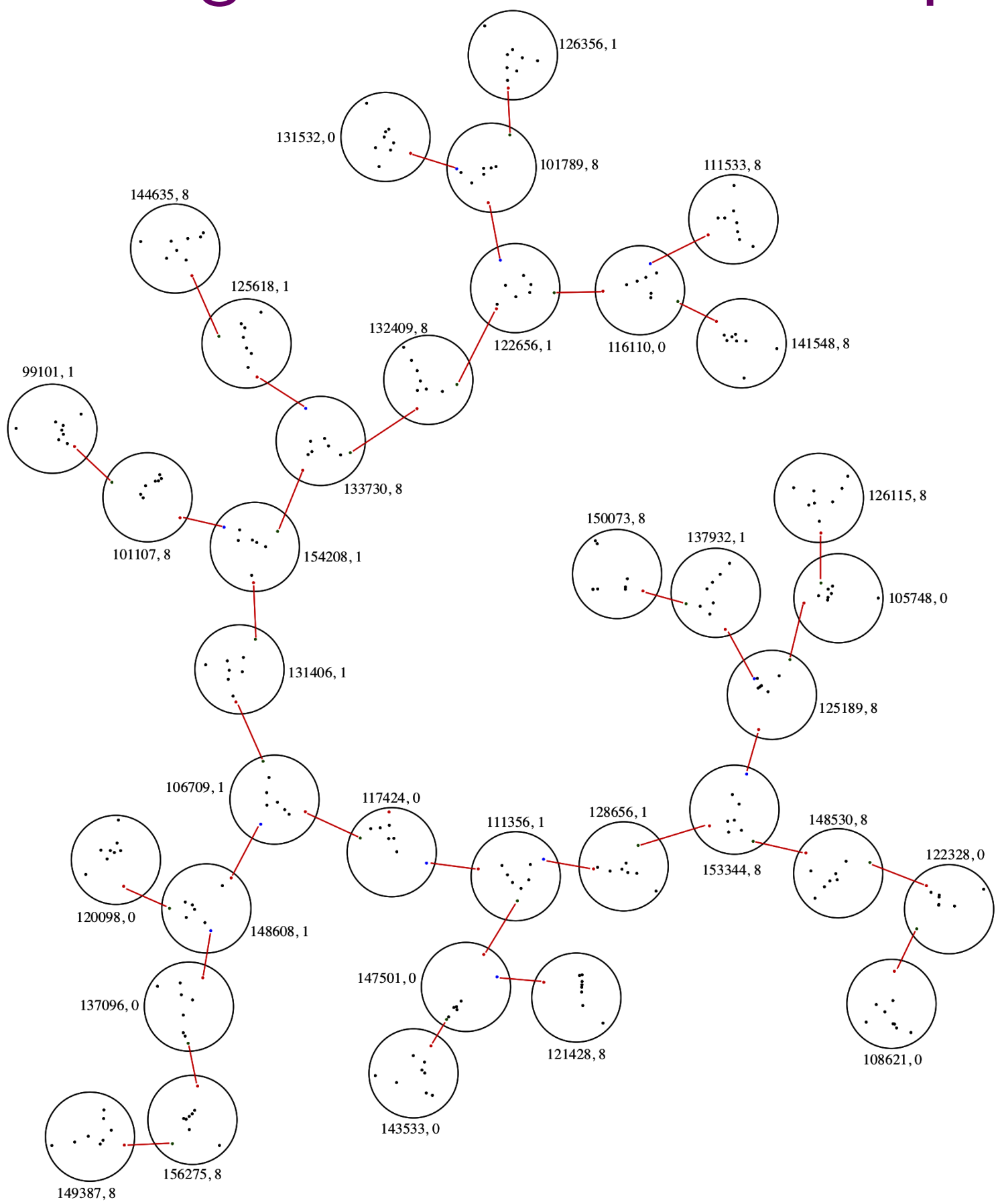
$\alpha_{x^*} \bowtie_{y^*} \beta$ is a realizable chirotope if and only if α and β are realizable and x^* and y^* are **extreme** in α and β . [Bouvel, Féray, Goaoc, K.]

\bowtie is associative: we can iterate the process to build a whole chirotope tree

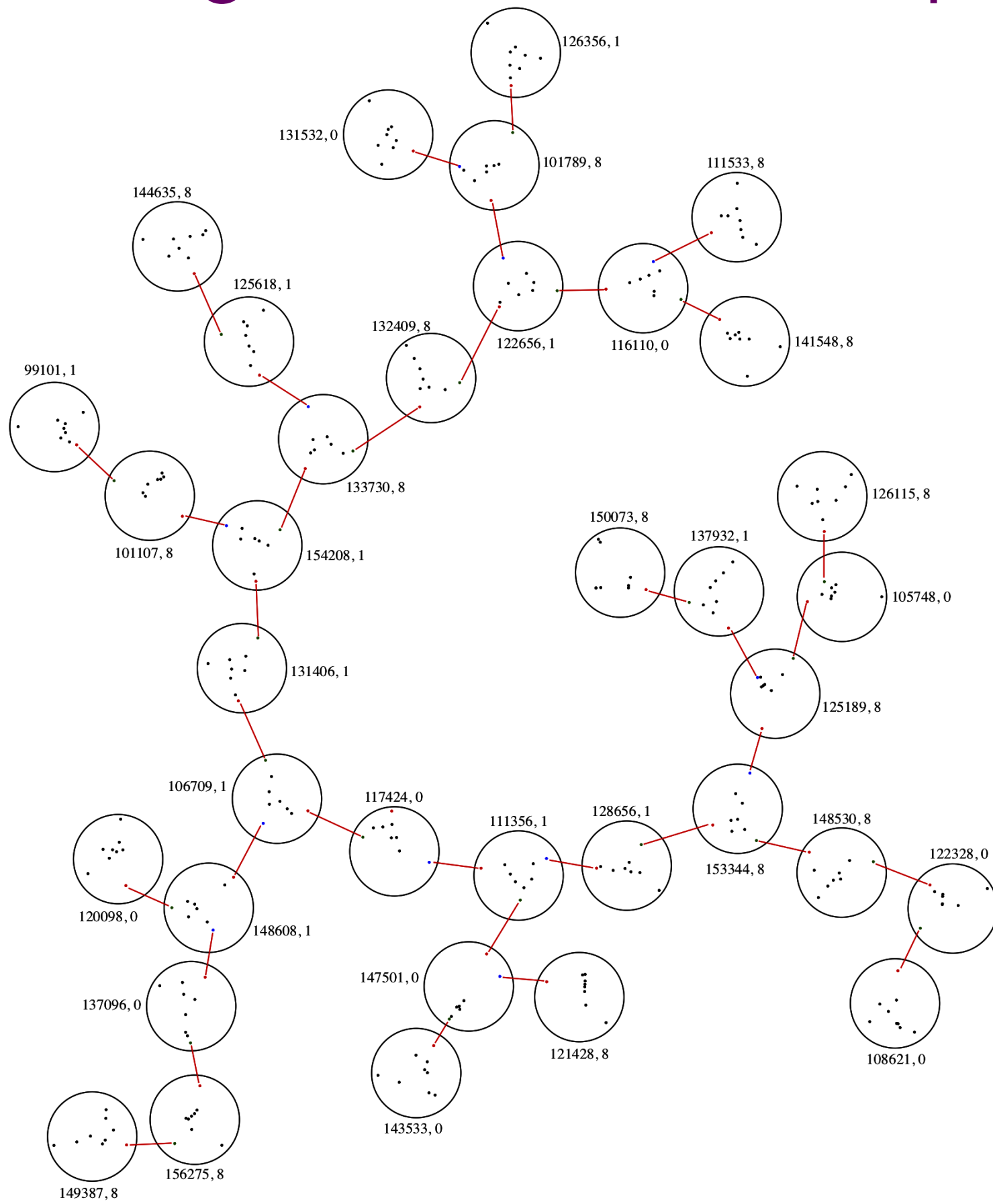
Triangulations of a chirotope tree

36 nodes, each decorated with a realizable chirotope of size 9, adding up to 254 elements.

Can we compute its number of triangulations?



Triangulations of a chirotope tree



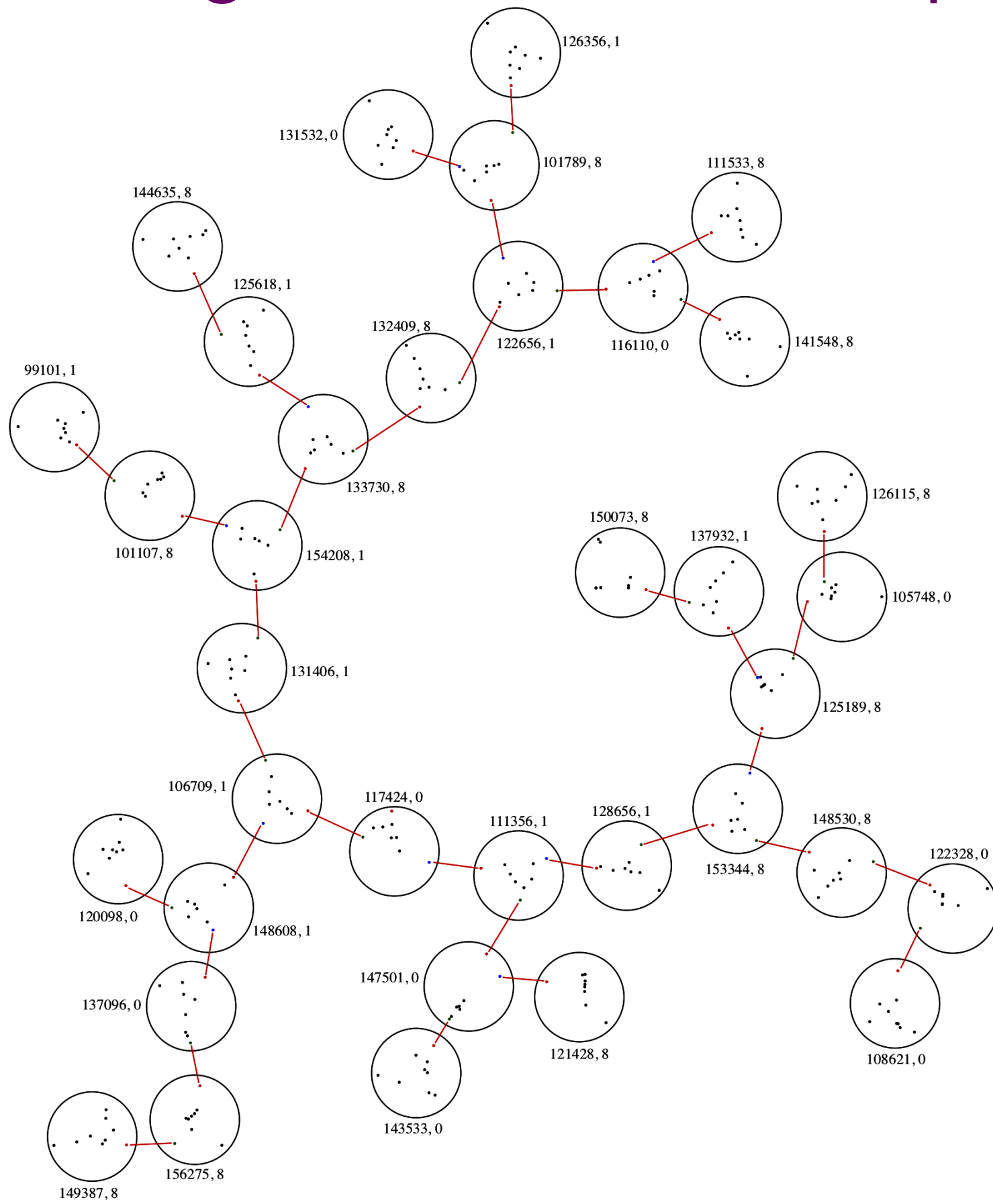
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Many questions on triangulations are still **open**:

- For every κ on n elements,
 - $|\mathcal{T}_\kappa| = \mathcal{O}(30^n)$ [Sharir and Sheffer, 2011]
 - $|\mathcal{T}_\kappa| = \Omega(2.63^n)$ [Aichholzer et al, 2016]
- It is conjectured that the minimal is $\mathcal{O}(3.47^n)$ [Hurtado and Noy 97]
- Max reached: Koch chains has $\approx 9.08^n$ triangulations [Rutschmann and Wettstein 2022]

Triangulations of a chirotope tree



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Fastest generic algorithms: Compute the number of triangulations of a given point set:

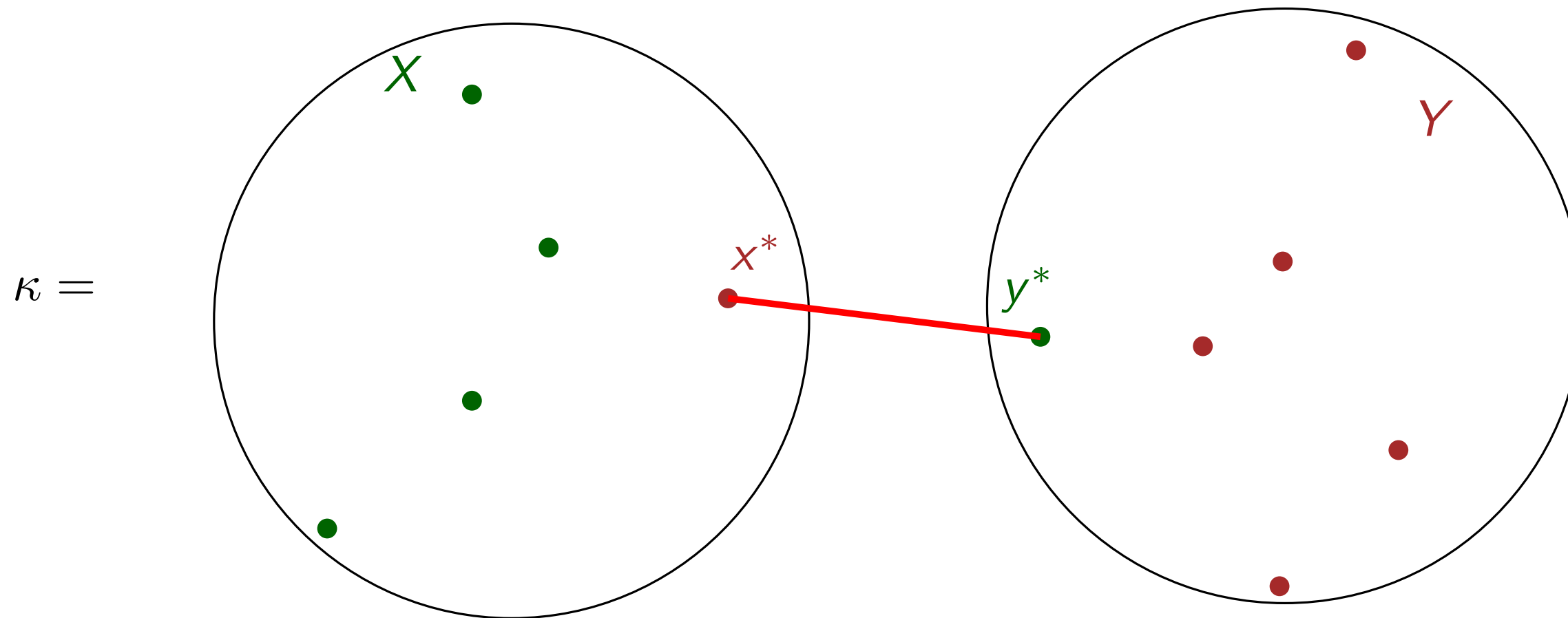
- $\mathcal{O}(n^2 2^n)$ [Alvarez and Seidel 2013]
- $\mathcal{O}(n^{(11+o(1))\sqrt{n}})$ [Marx and Miltzow 2016]

Strategy Combine generic algorithms with bowtie properties

Triangulations of a bowtie

α realizable chirotope on $X \cup \{x^*\}$, and β on $Y \cup \{y^*\}$

$$\kappa = \alpha_{x^*} \bowtie_{y^*} \beta$$



Triangulations of a bowtie

α realizable chirotope on $X \cup \{x^*\}$, and β on $Y \cup \{y^*\}$

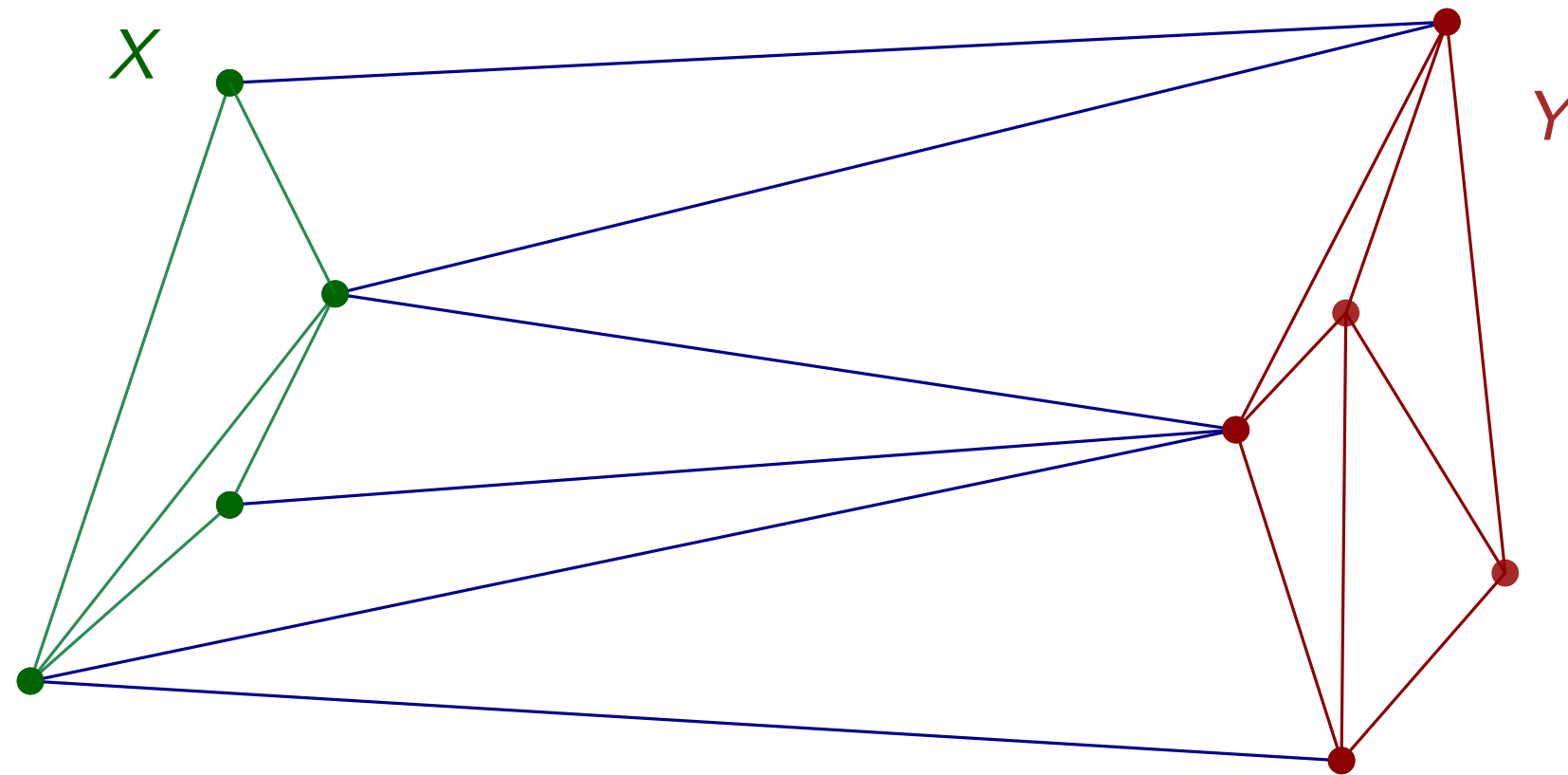
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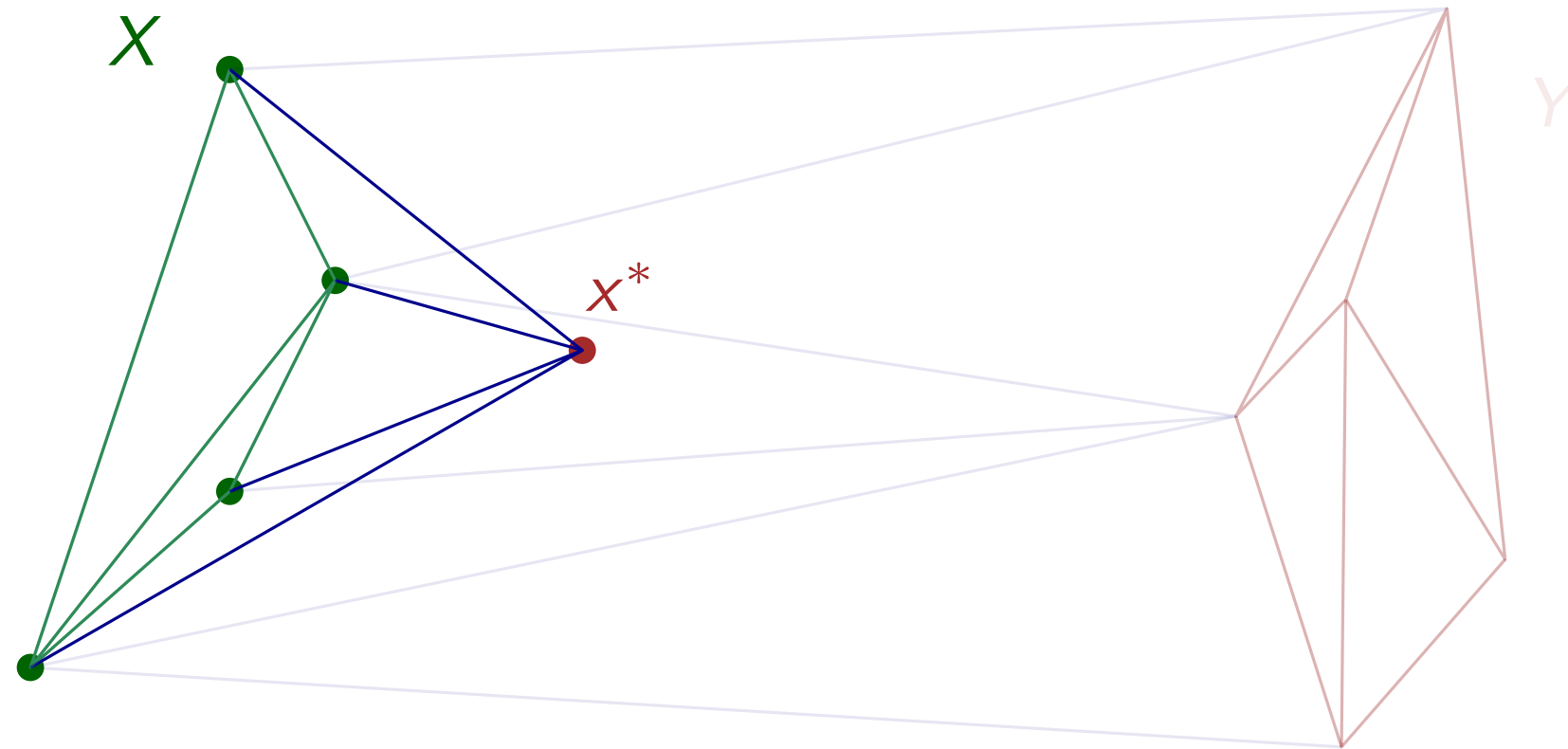


Triangulations of a bowtie

α realizable chirotope on $X \cup \{x^*\}$, and β on $Y \cup \{y^*\}$

$$\kappa = \alpha_{x^*} \bowtie_{y^*} \beta$$

$$T' = \pi_{Y \rightarrow x^*}(T)$$

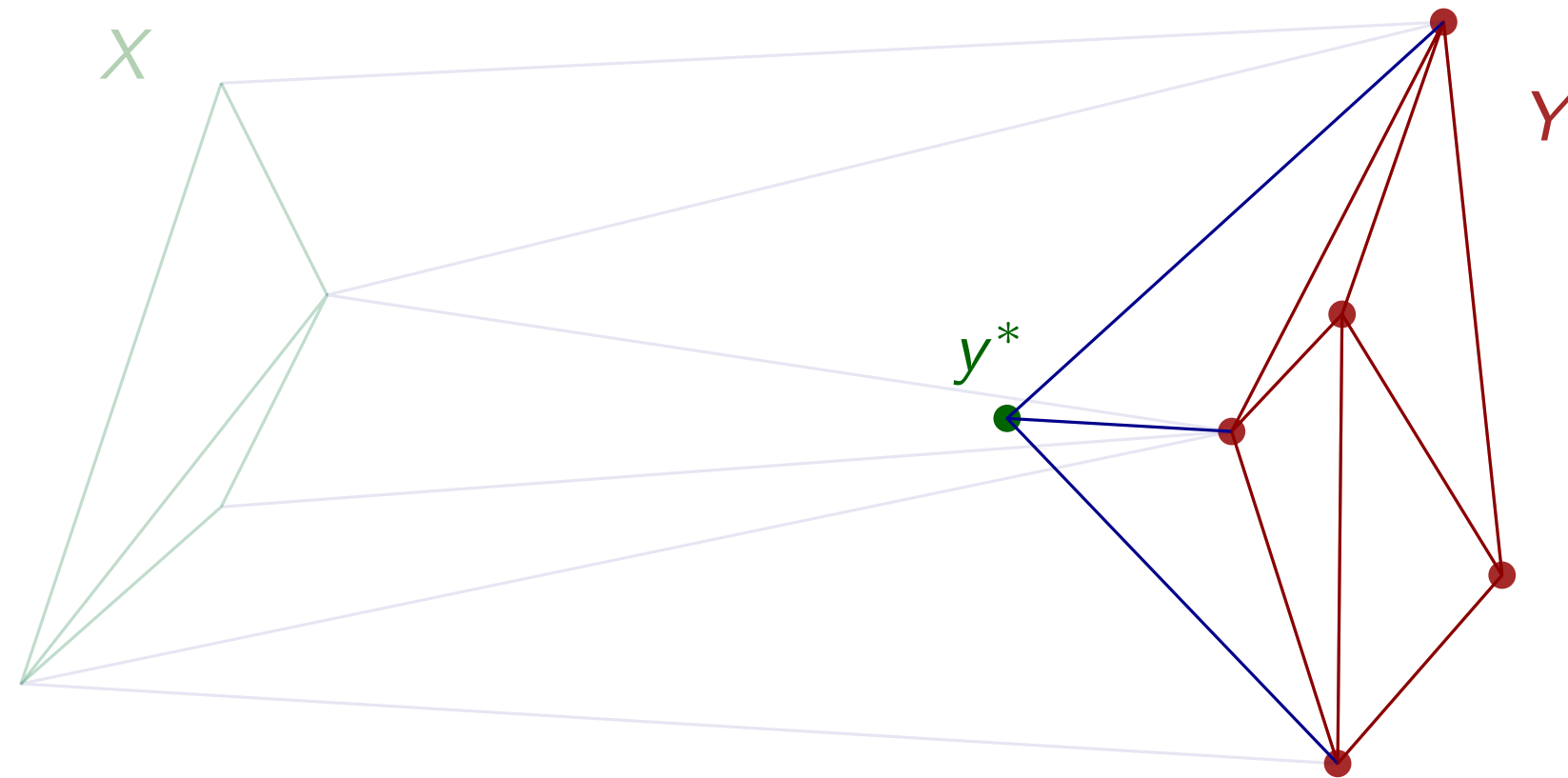


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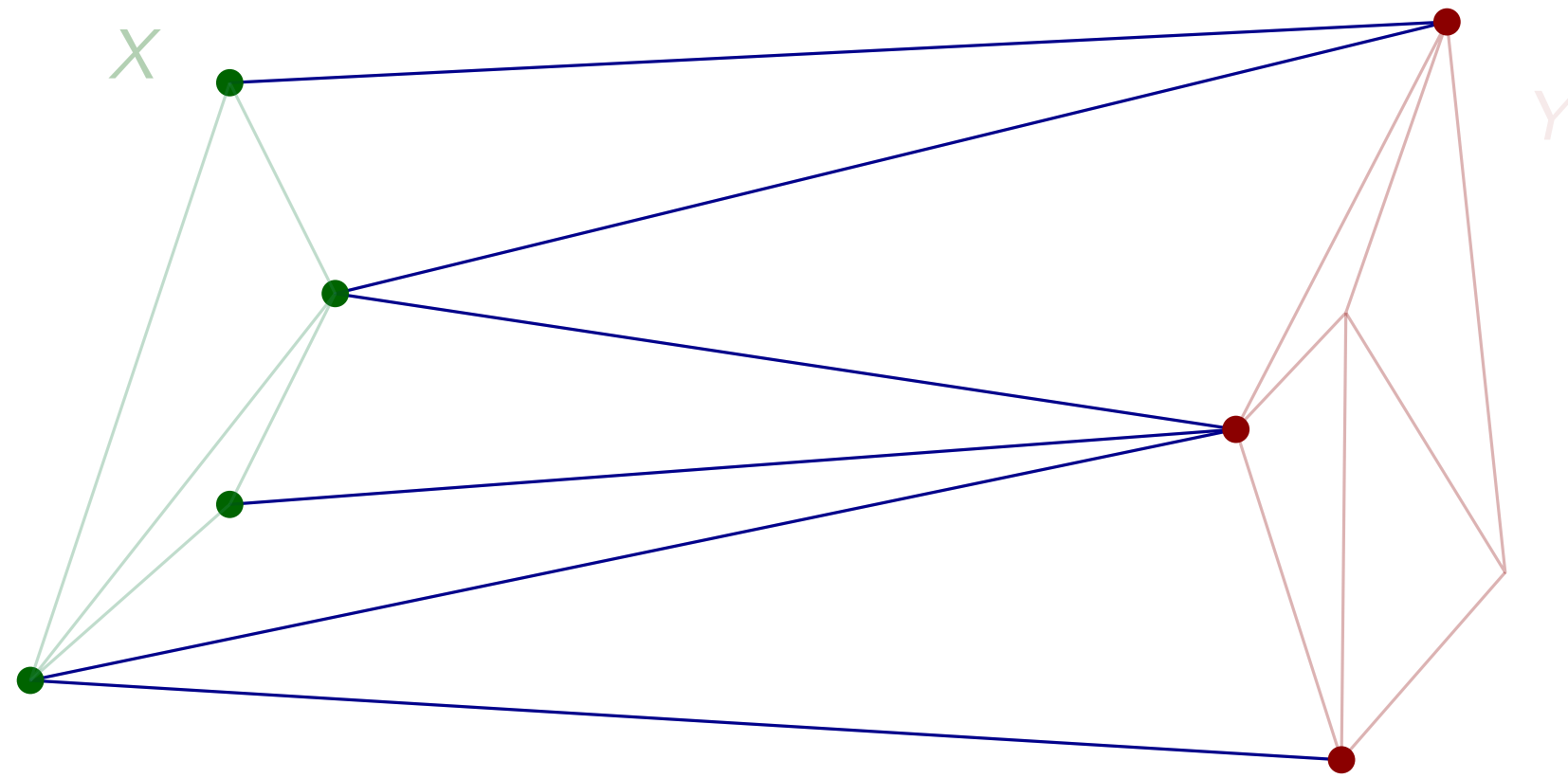


Triangulations of a bowtie

α realizable chirotope on $X \cup \{x^*\}$, and β on $Y \cup \{y^*\}$

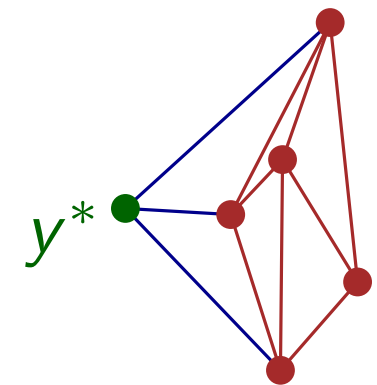
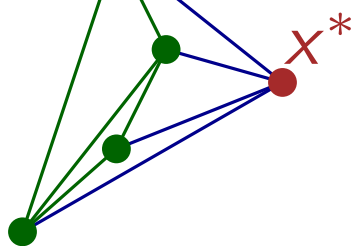
$$\kappa = \alpha_{x^*} \bowtie_{y^*} \beta$$

T_{XY}



$\deg_{T'}(x^*)$

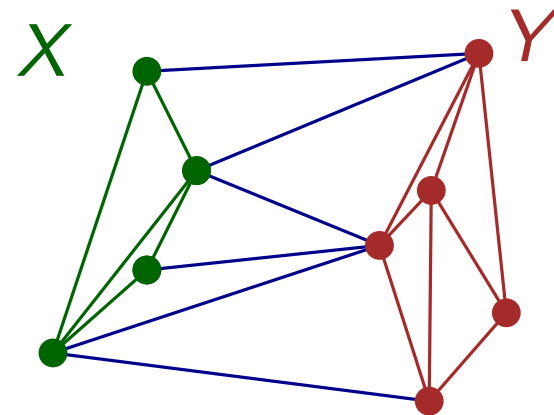
$\deg_{T''}(y^*)$



Triangulations of a bowtie

α realizable chirotope on $X \cup \{x^*\}$, and β on $Y \cup \{y^*\}$

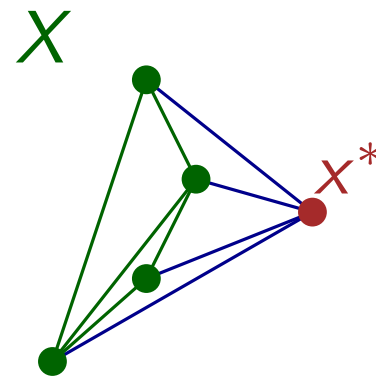
$$\kappa = \alpha_{x^*} \bowtie_{y^*} \beta$$



$$T \in \mathcal{T}_\kappa$$

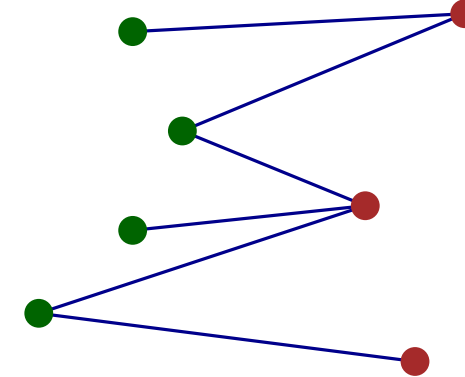
Triangulations of κ

Bijection

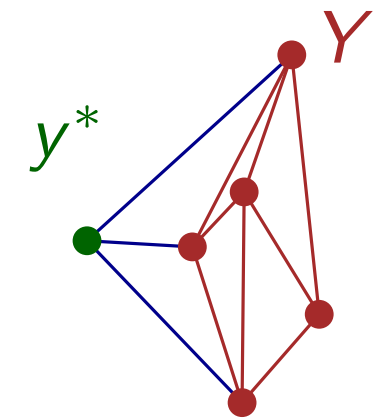


$$T' = \pi_{Y \rightarrow x^*}(T)$$

$\deg_{T'}(x^*)$ $\deg_{T''}(y^*)$



$$T_{XY}$$



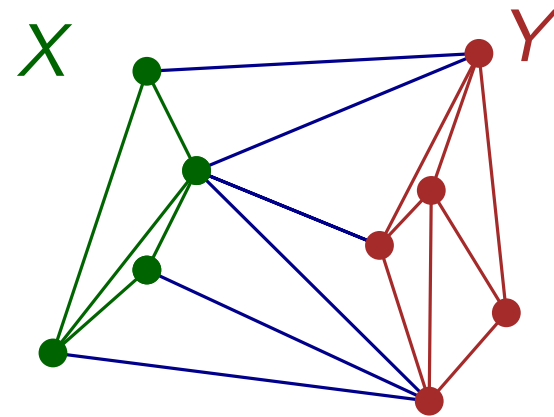
$$T'' = \pi_{X \rightarrow y^*}(T)$$

Triplets of triangulations of α , β , and maximal crossing-free families of edges between the neighbors of x^* and y^*

Triangulations of a bowtie

α realizable chirotope on $X \cup \{x^*\}$, and β on $Y \cup \{y^*\}$

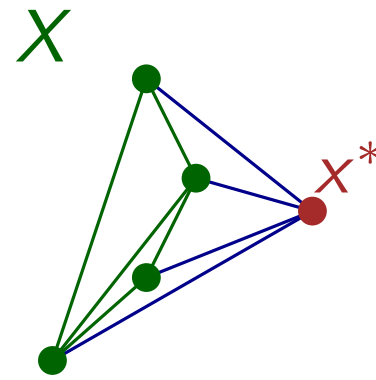
$$\kappa = \alpha_{x^*} \bowtie_{y^*} \beta$$



$$T \in \mathcal{T}_\kappa$$

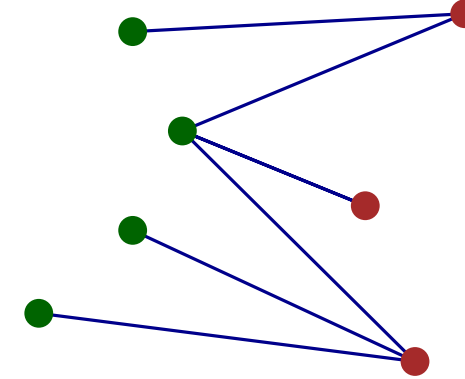
Triangulations of κ

Bijection

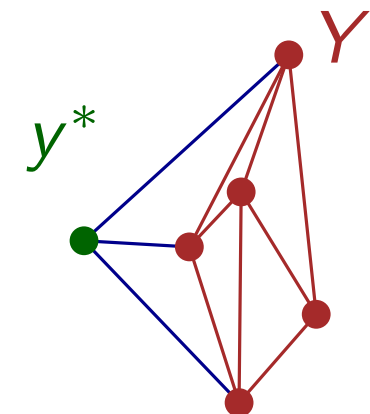


$$T' = \pi_{Y \rightarrow x^*}(T)$$

$\deg_{T'}(x^*)$ $\deg_{T''}(y^*)$



$$T_{XY}$$



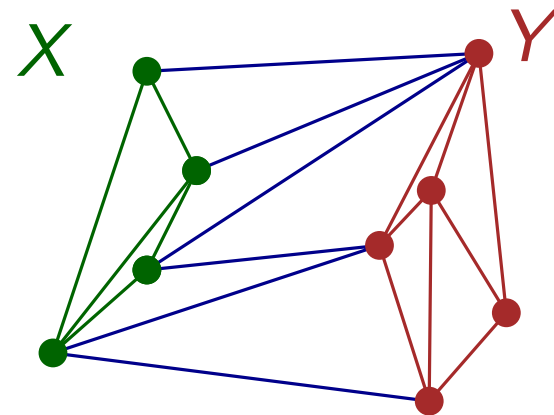
$$T'' = \pi_{X \rightarrow y^*}(T)$$

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Triangulations of a bowtie

α realizable chirotope on $X \cup \{x^*\}$, and β on $Y \cup \{y^*\}$

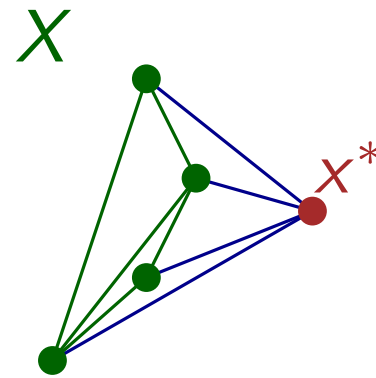
$$\kappa = \alpha_{x^*} \bowtie_{y^*} \beta$$



$$T \in \mathcal{T}_\kappa$$

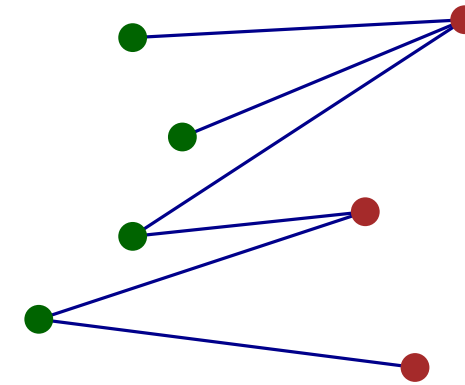
Triangulations of κ

Bijection

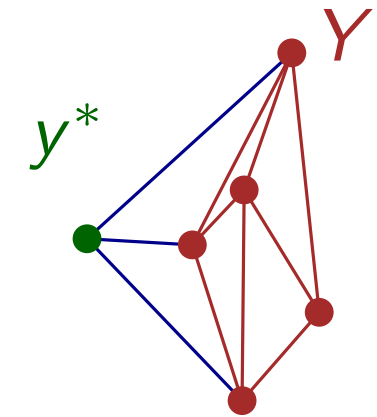


$$T' = \pi_{Y \rightarrow x^*}(T)$$

$\deg_{T'}(x^*)$ $\deg_{T''}(y^*)$



$$T_{XY}$$



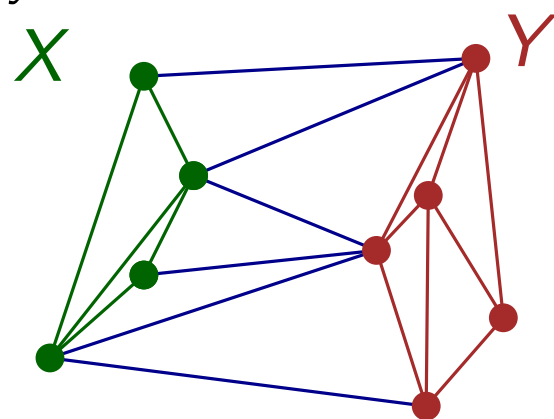
$$T'' = \pi_{X \rightarrow y^*}(T)$$

Triplets of triangulations of α , β , and maximal crossing-free families of edges between the neighbors of x^* and y^*

Triangulations of a bowtie

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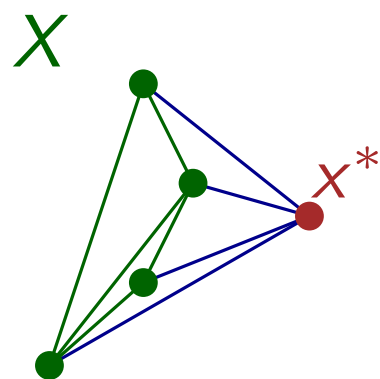
$$\kappa = \alpha_{x^*} \bowtie_{y^*} \beta$$



$$T \in \mathcal{T}_\kappa$$

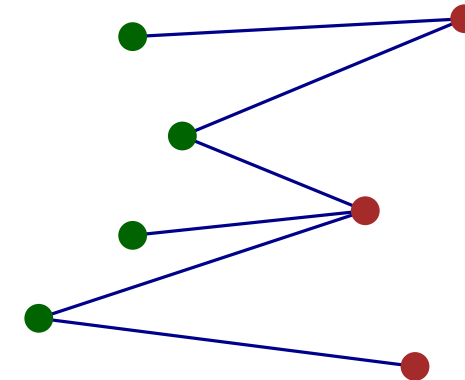
Triangulations of κ

Bijection

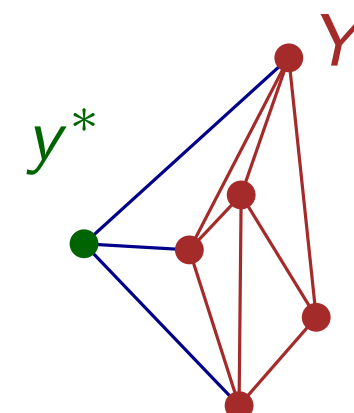


$$T' = \pi_{Y \rightarrow x^*}(T)$$

$\deg_{T'}(x^*)$ $\deg_{T''}(y^*)$



$$T_{XY}$$



$$T'' = \pi_{X \rightarrow y^*}(T)$$

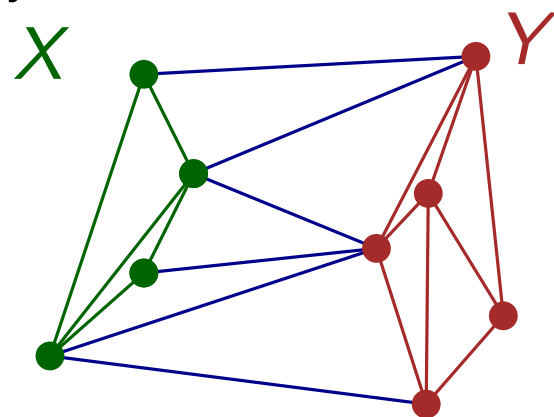
Triplets of triangulations of α , β , and maximal crossing-free families of edges between the neighbors of x^* and y^*

$$|\mathcal{T}_\kappa| = \sum_{a,b \geq 2} \binom{a+b-2}{a-1} [x^a] P_{\alpha, x^*}(x) [y^b] P_{\beta, y^*}(y)$$

Triangulations of a bowtie

α realizable chirotope on $X \cup \{x^*\}$, and β on $Y \cup \{y^*\}$

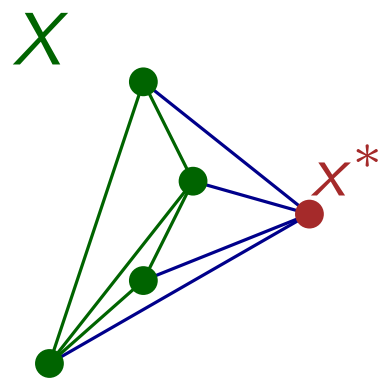
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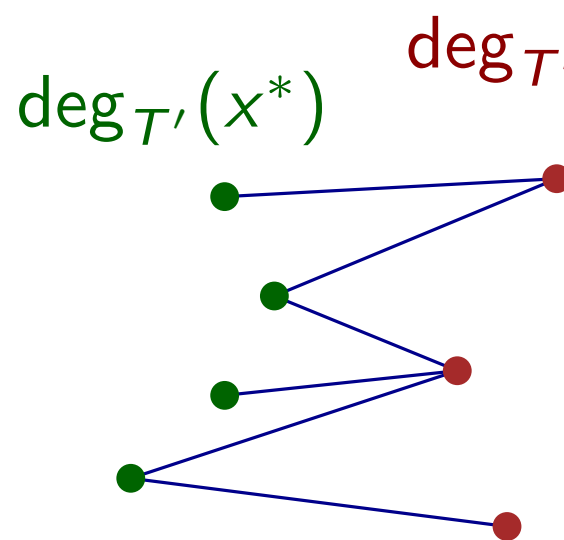
$$T \in \mathcal{T}_\kappa$$

Triangulations of κ

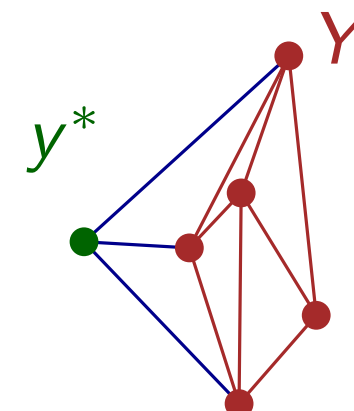
Bijection



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$$T_{XY}$$



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Triangulations of a bowtie

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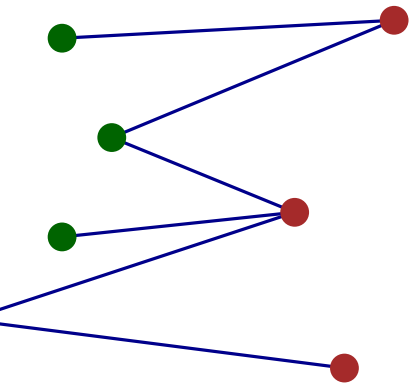
$\kappa = \alpha$

a

b



$$a = \deg_{T'}(x^*)$$



$$b = \deg_{T''}(y^*) \quad (T)$$

g-free
 y^*

$$|\mathcal{T}_\kappa| = \sum_{a,b \geq 2} \binom{a+b-2}{a-1} [x^a] P_{\alpha, x^*}(x) [y^b] P_{\beta, y^*}(y)$$

Triangulations of a bowtie

α realizable chirotope on $X \cup \{x^*\}$ and β on $Y \cup \{y^*\}$

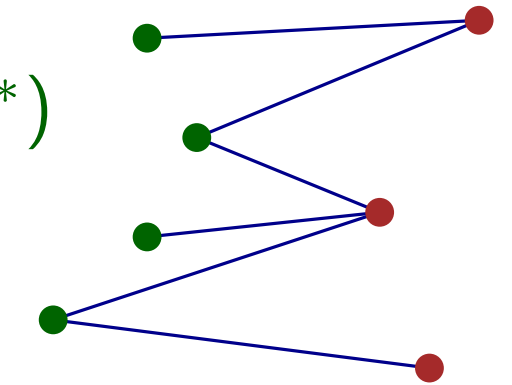
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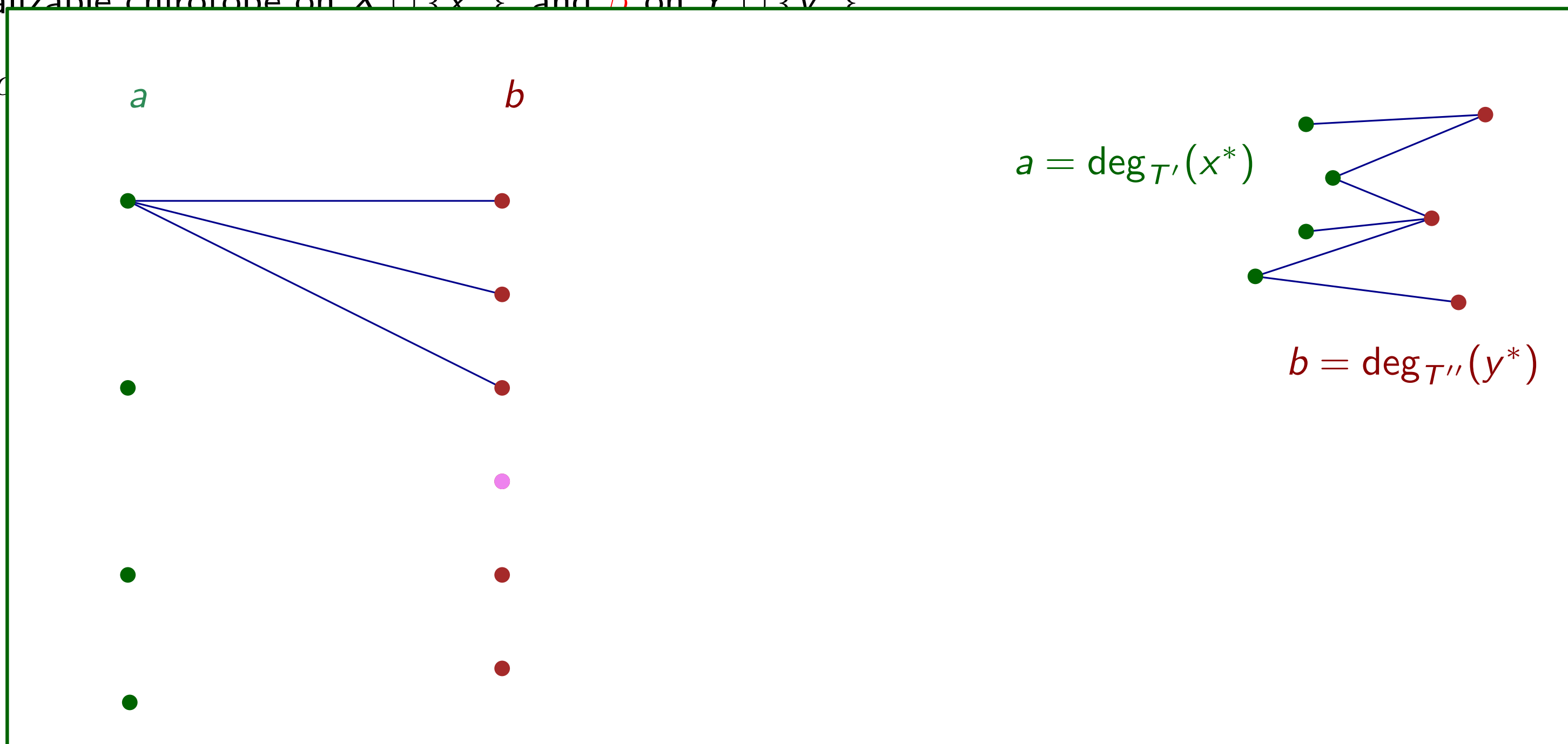
g-free
 y^*

$$|\mathcal{T}_\kappa| = \sum_{a,b \geq 2} \binom{a+b-2}{a-1} [x^a] P_{\alpha, x^*}(x) [y^b] P_{\beta, y^*}(y)$$

Triangulations of a bowtie

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$\kappa = \alpha$

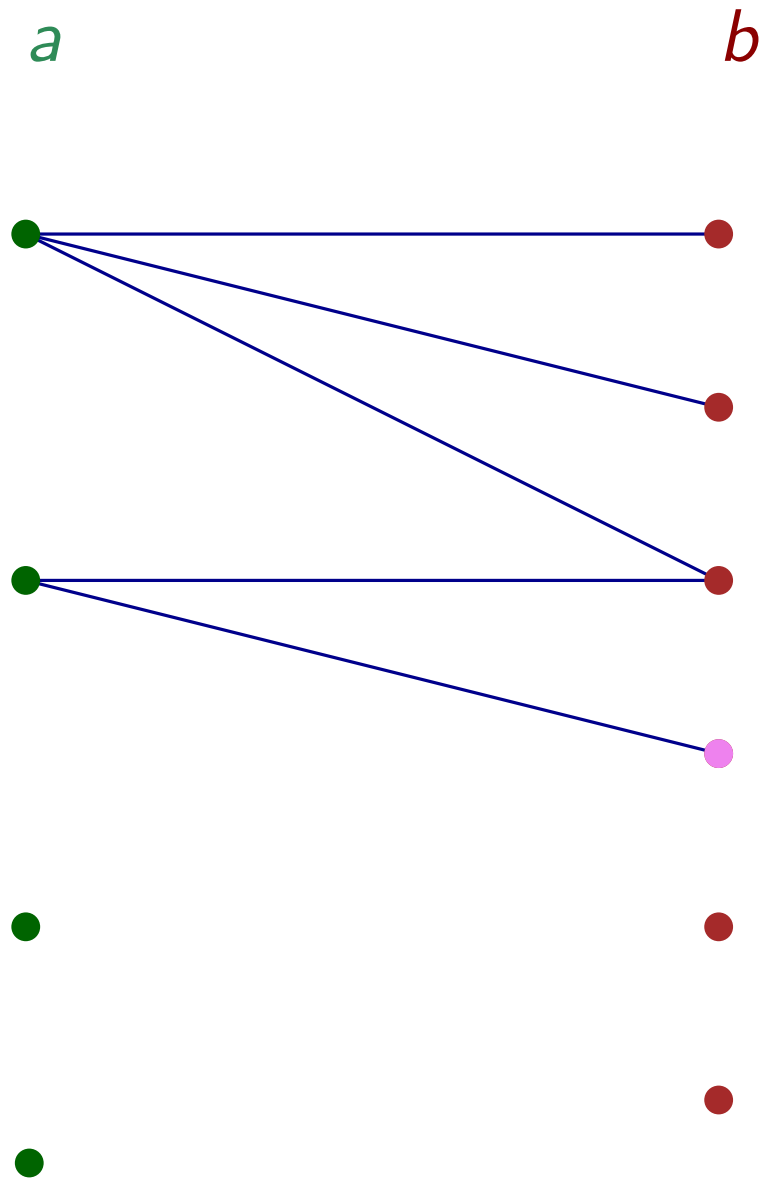


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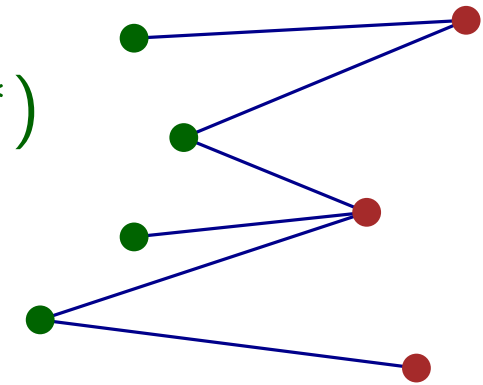
Triangulations of a bowtie

α realizable chirotope on $X \cup \{x^*\}$ and β on $Y \cup \{y^*\}$

$\kappa = \alpha$



$$a = \deg_{T'}(x^*)$$



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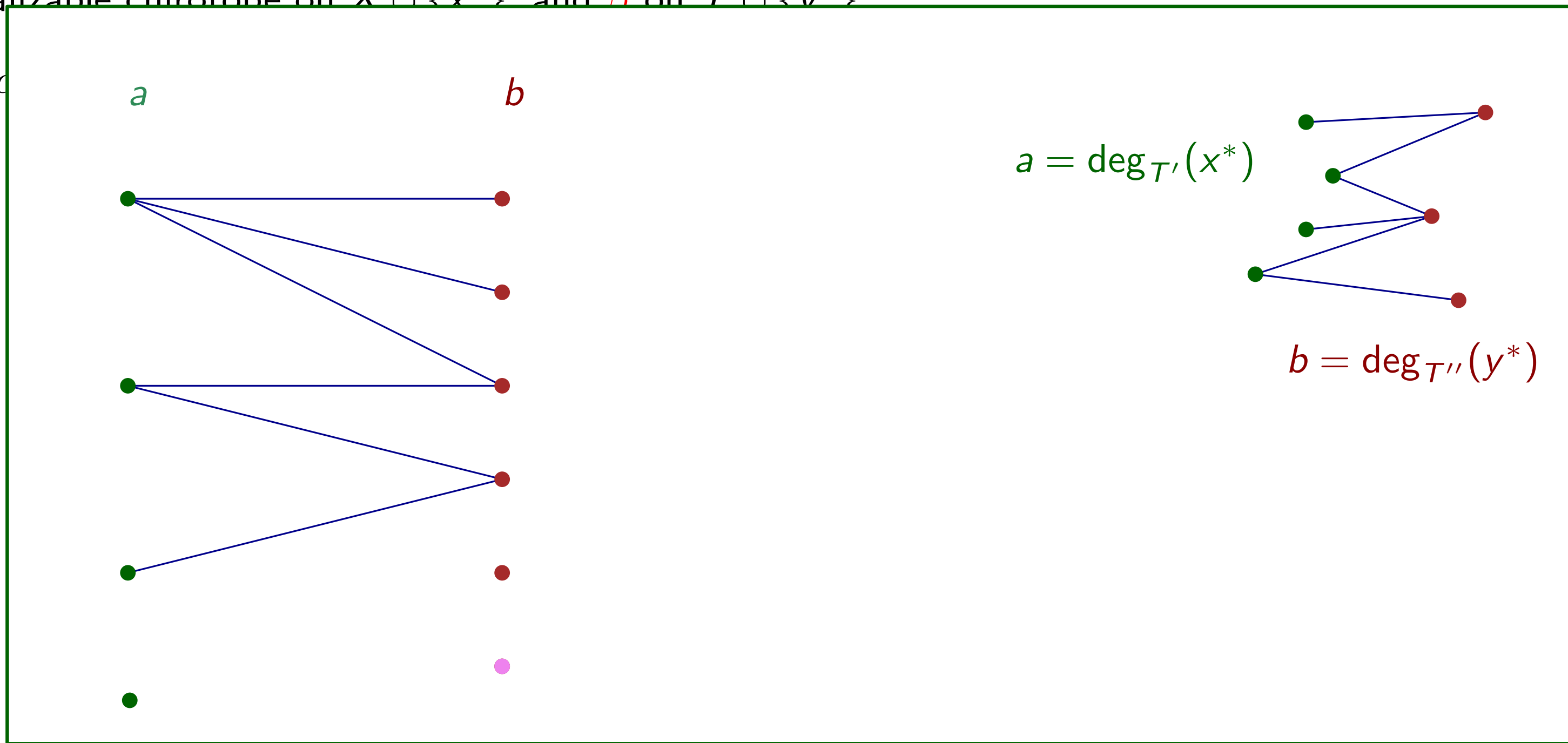
g-free
 y^*

$$|\mathcal{T}_\kappa| = \sum_{a,b \geq 2} \binom{a+b-2}{a-1} [x^a] P_{\alpha, x^*}(x) [y^b] P_{\beta, y^*}(y)$$

Triangulations of a bowtie

α realizable chirotope on $X \cup \{x^*\}$ and β on $Y \cup \{y^*\}$

$\kappa = 0$

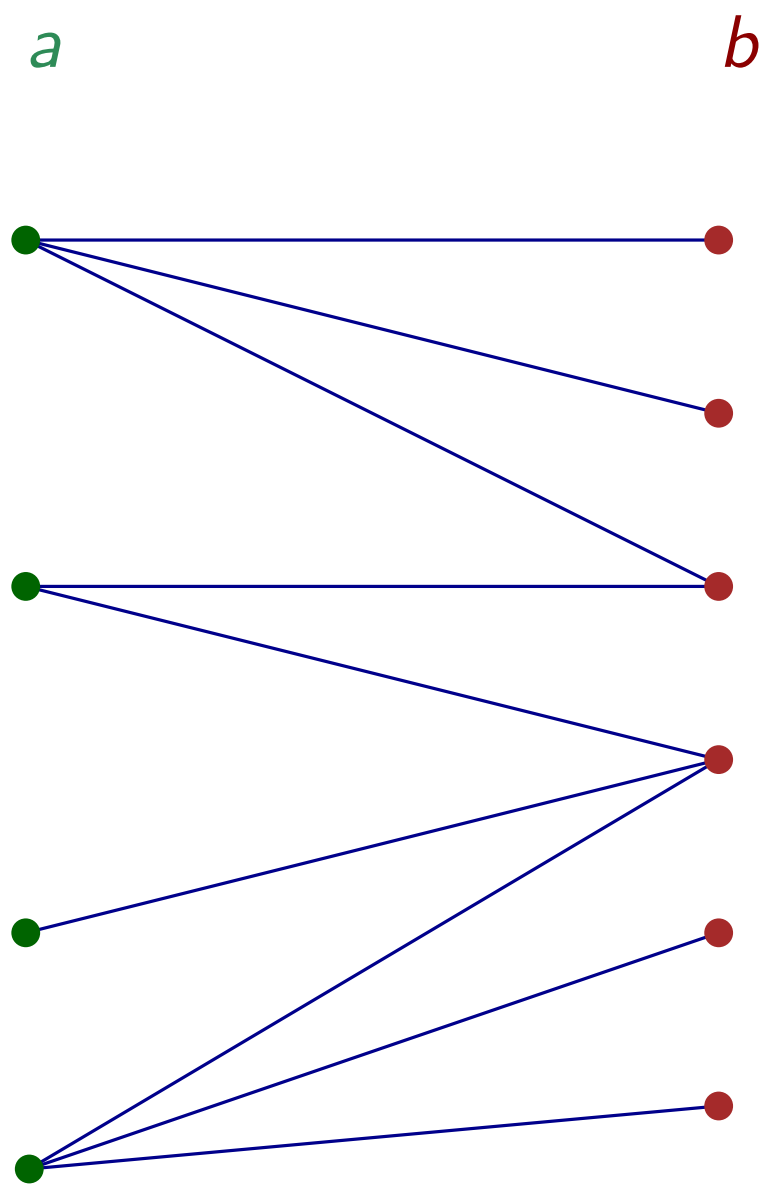


$$|\mathcal{T}_\kappa| = \sum_{a,b \geq 2} \binom{a+b-2}{a-1} [x^a] P_{\alpha, x^*}(x) [y^b] P_{\beta, y^*}(y)$$

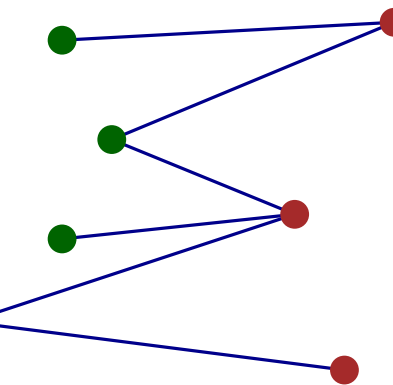
Triangulations of a bowtie

α realizable chirotope on $X \cup \{x^*\}$ and β on $Y \cup \{y^*\}$

$\kappa = \alpha$



$$a = \deg_{T'}(x^*)$$



$$b = \deg_{T''}(y^*) \quad (T)$$

g-free
 y^*

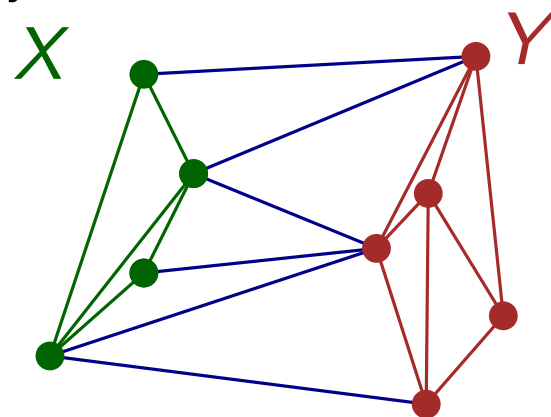
$$|T_{XY}| = \binom{b}{a-1} = \binom{a+b-2}{a-1}$$

$$|\mathcal{T}_\kappa| = \sum_{a,b \geq 2} \binom{a+b-2}{a-1} [x^a] P_{\alpha, x^*}(x) [y^b] P_{\beta, y^*}(y)$$

Triangulations of a bowtie

α realizable chirotope on $X \cup \{x^*\}$, and β on $Y \cup \{y^*\}$

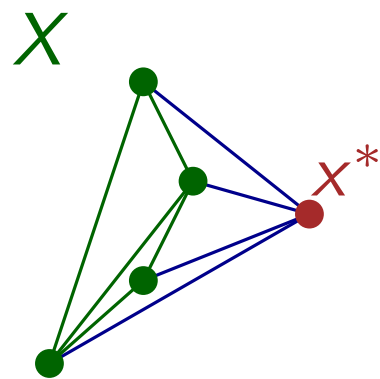
$$\kappa = \alpha_{x^*} \bowtie_{y^*} \beta$$



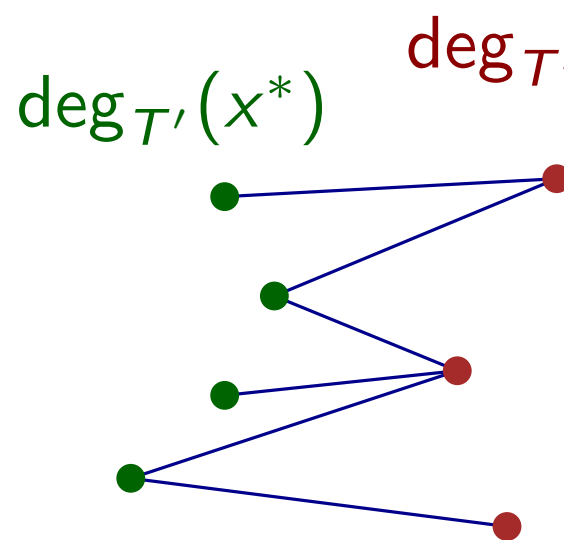
$$T \in \mathcal{T}_\kappa$$

Triangulations of κ

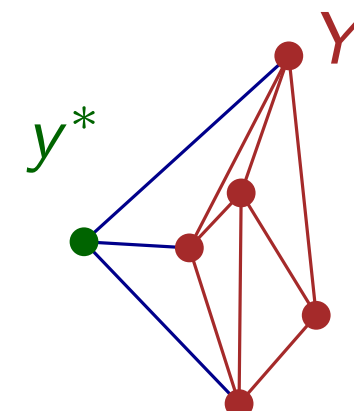
Bijection



$$T' = \pi_{Y \rightarrow x^*}(T)$$



$$T_{XY}$$



$$T'' = \pi_{X \rightarrow y^*}(T)$$

Triples of triangulations of α , β , and maximal crossing-free families of edges between the neighbors of x^* and y^*

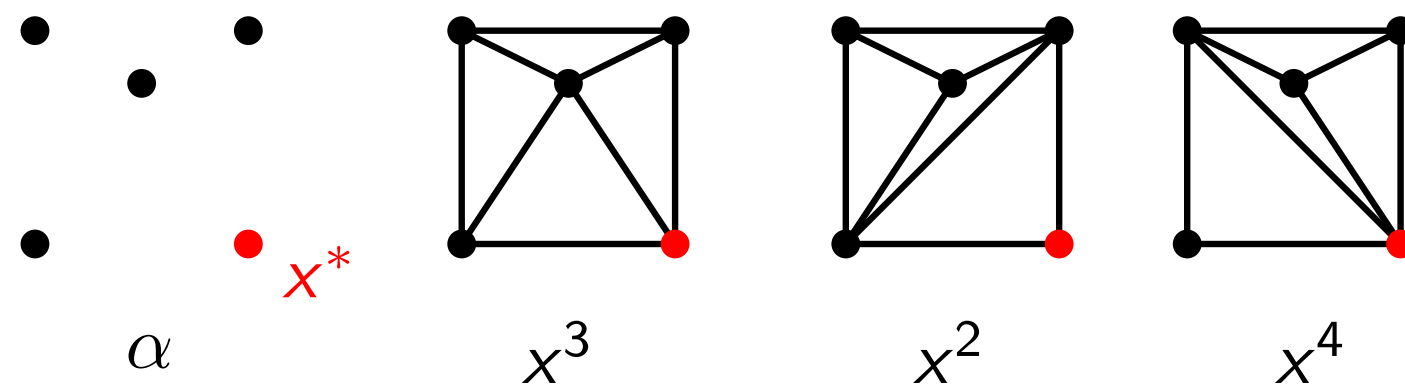
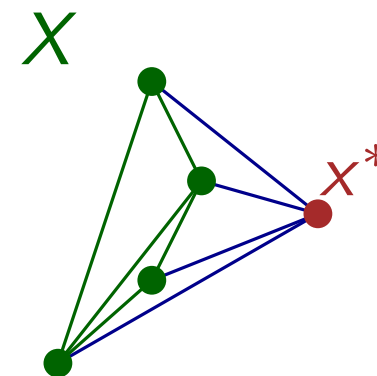
$$|\mathcal{T}_\kappa| = \sum_{a,b \geq 2} \binom{a+b-2}{a-1} [x^a] P_{\alpha, x^*}(x) [y^b] P_{\beta, y^*}(y)$$

Triangulations of a bowtie

α realizable chirotope on $X \cup \{x^*\}$ and β on $Y \cup \{y^*\}$

$$P_{\alpha, x^*}(x) = \sum_{T \in \mathcal{T}_\alpha} x^{\deg_T(x^*)} \text{ triangulation polynomial}$$

$$P_{\alpha, x^*}(1) = |\mathcal{T}_\alpha|$$



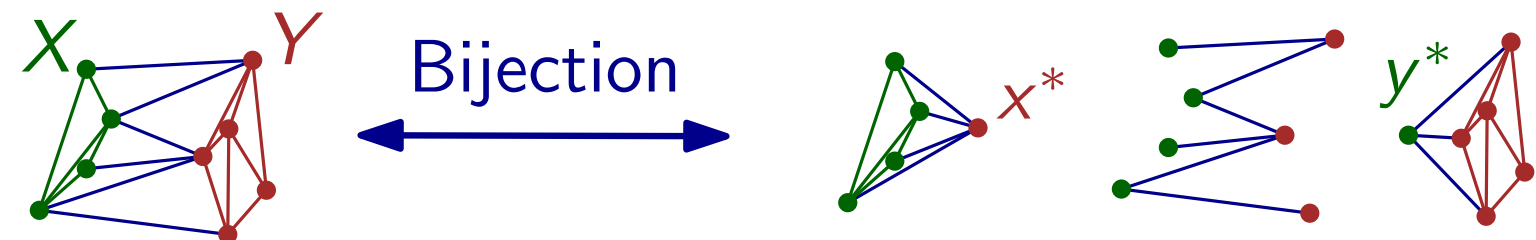
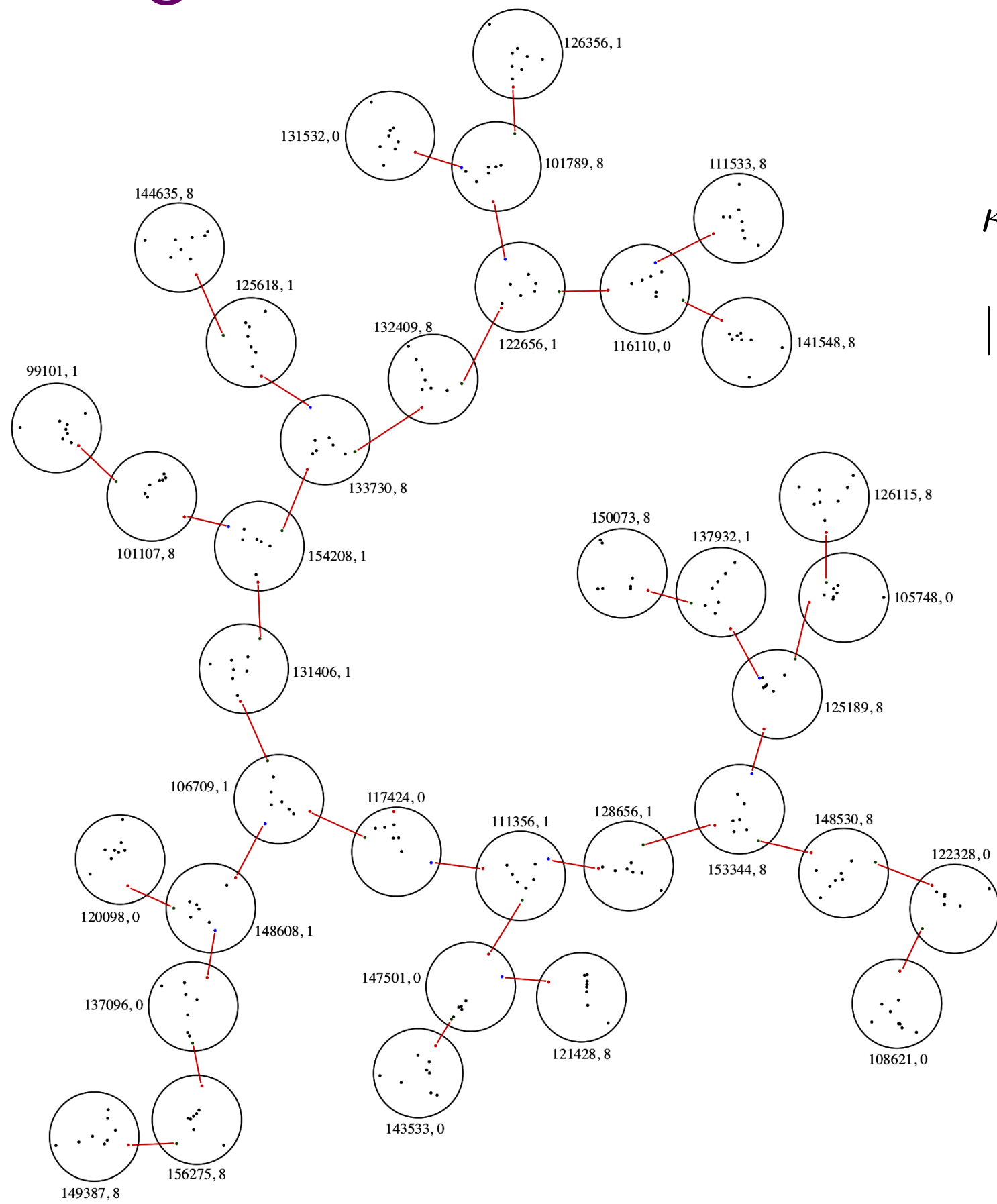
$$P_{\alpha, x^*}(x) = x^2(1 + x + x^2) \quad P_{\alpha, x^*}(1) = 3$$

$[x^a]P_{\alpha, x^*}(x)$ = number of triangulations of α where $\deg(x) = a$

$$|\mathcal{T}_\kappa| = \sum_{a, b \geq 2} \binom{a + b - 2}{a - 1} [x^a]P_{\alpha, x^*}(x) [y^b]P_{\beta, y^*}(y)$$

(T)
g-free
y*

Triangulations of a bowtie



$$\kappa = \alpha_{x^*} \bowtie_{y^*} \beta$$

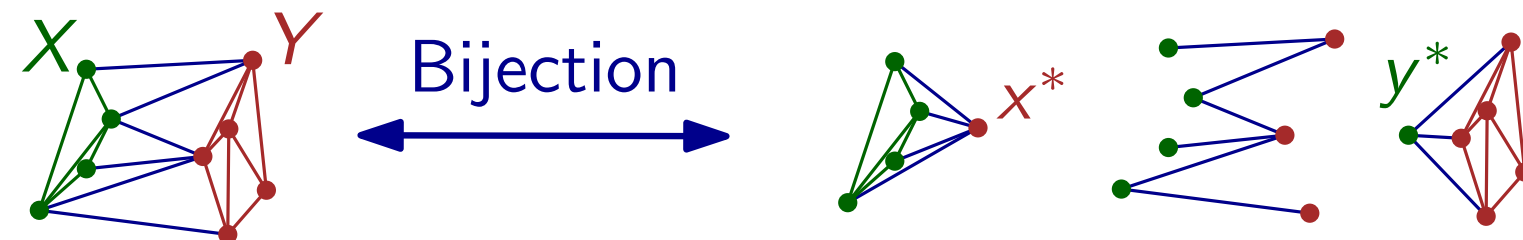
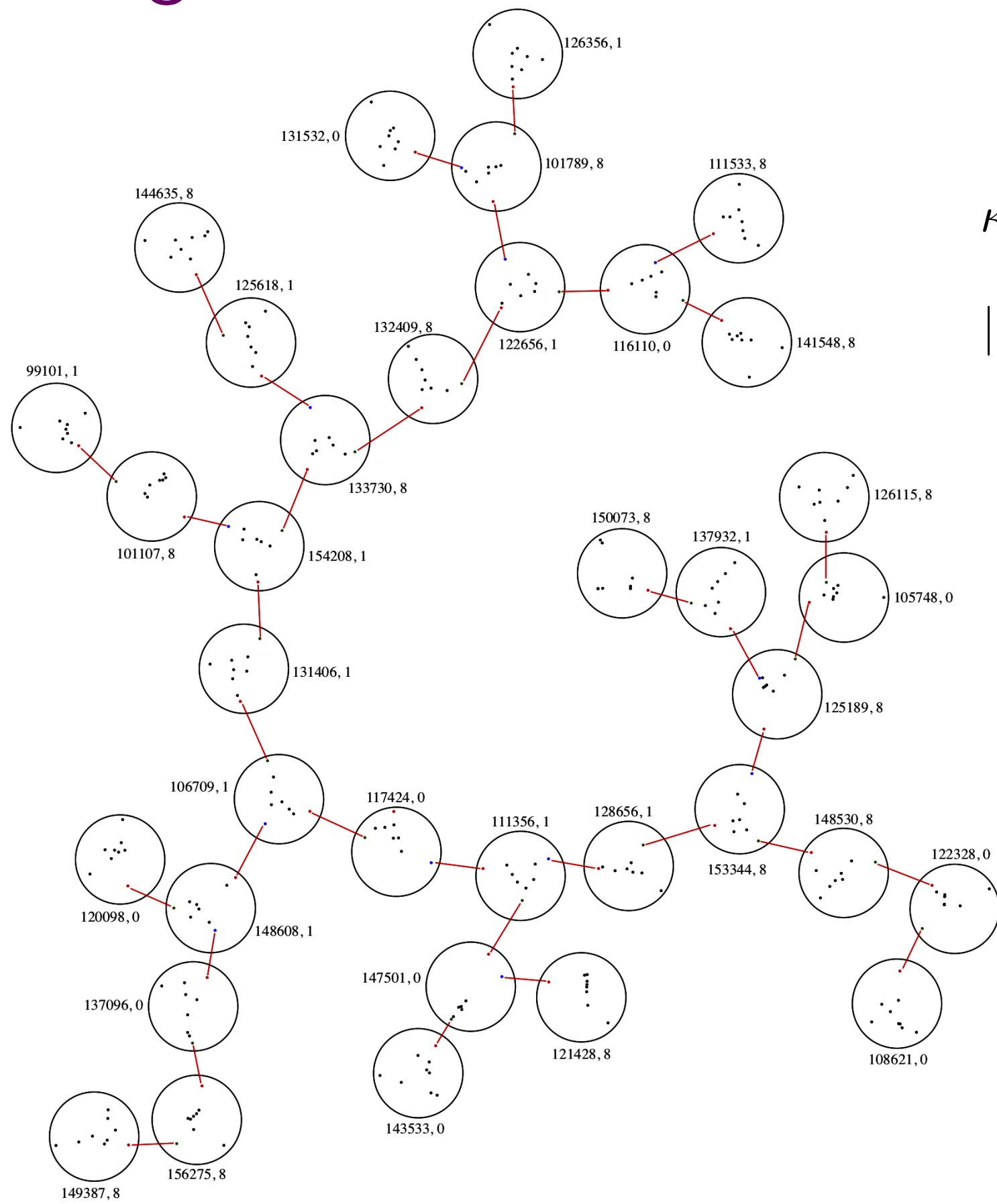
$$|\mathcal{T}_\kappa| = \sum_{a,b \geq 2} \binom{a+b-2}{a-1} [x^a] P_{\alpha, x^*}(x) [y^b] P_{\beta, y^*}(y)$$

36 nodes, each decorated with a chirotope of size 9, adding up to 254 elements.

Number of triangulations?

- Same general idea with the bijection
- Involves full triangulation polynomials
 - marking degrees of proxies
 - keeping track of edges between proxies

Triangulations of a bowtie



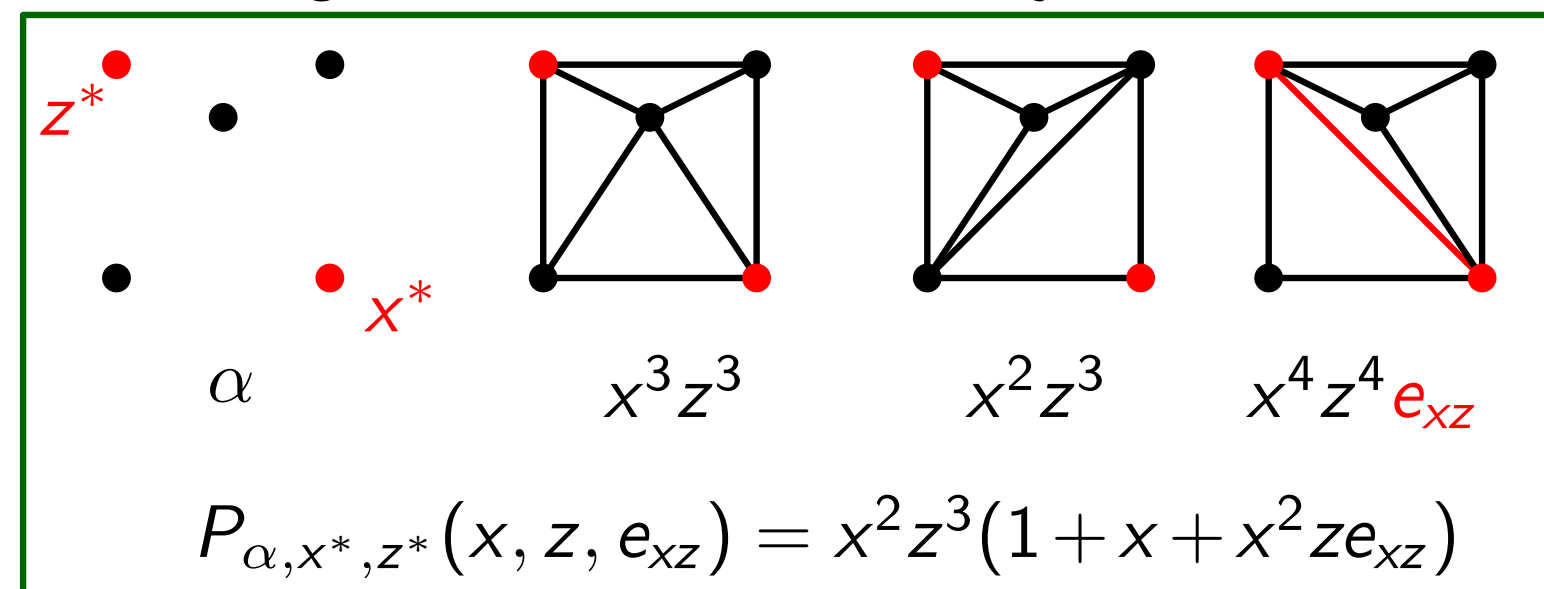
$$\kappa = \alpha_{x^*} \bowtie_{y^*} \beta$$

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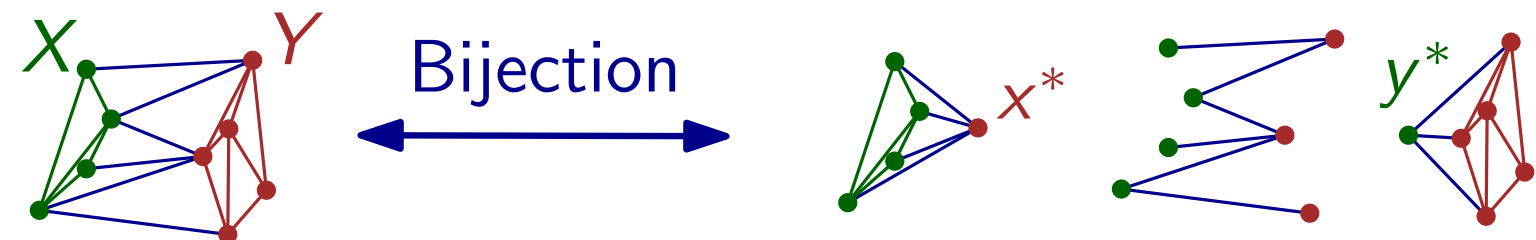
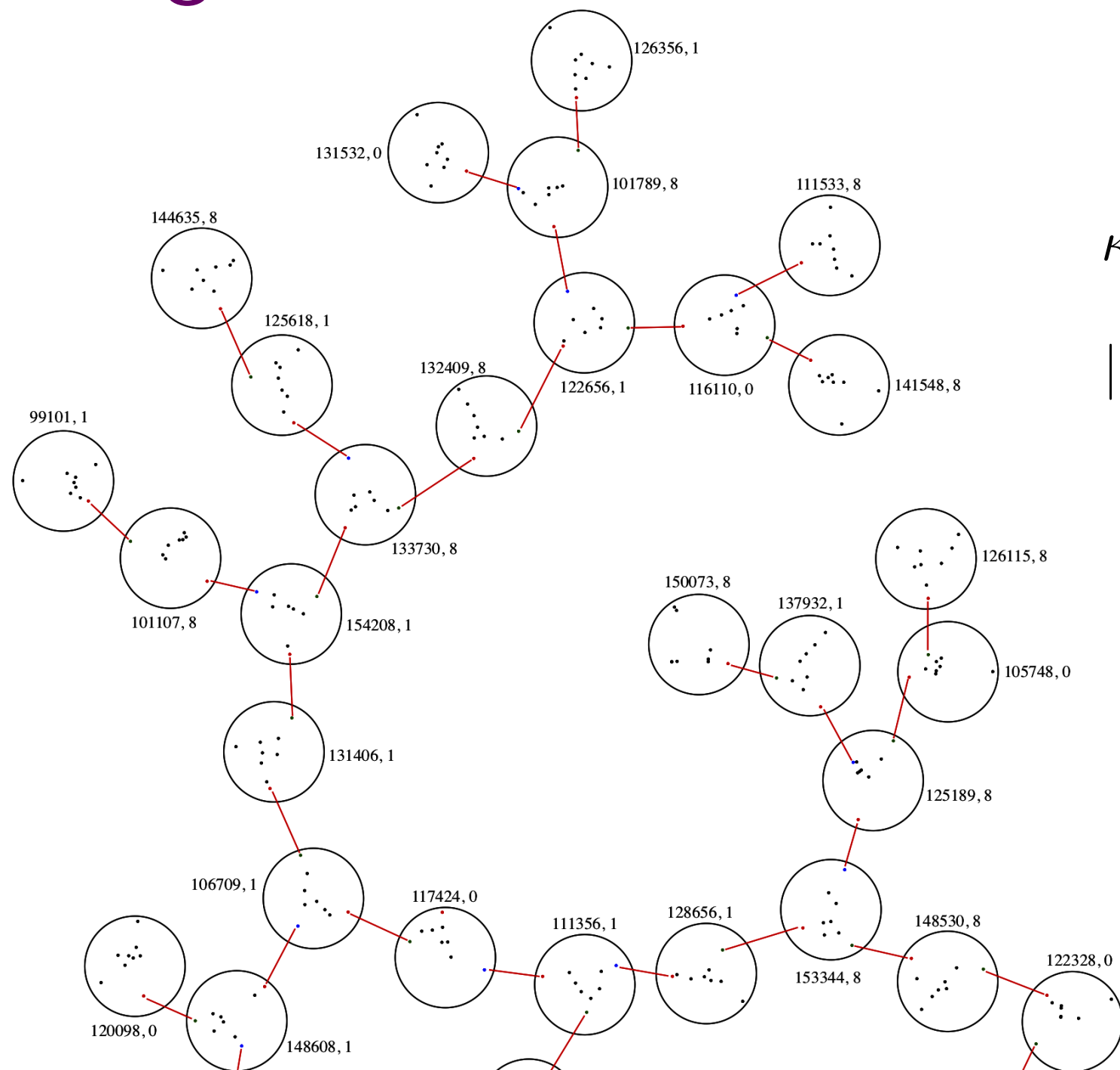
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Triangulations of a bowtie



$$\kappa = \alpha_{x^*} \bowtie_{y^*} \beta$$

$$|\mathcal{T}_\kappa| = \sum_{a,b \geq 2} \binom{a+b-2}{a-1} [x^a] P_{\alpha, x^*}(x) [y^b] P_{\beta, y^*}(y)$$

36 nodes, each decorated with a chirotope of size 9, adding up to 254 elements.

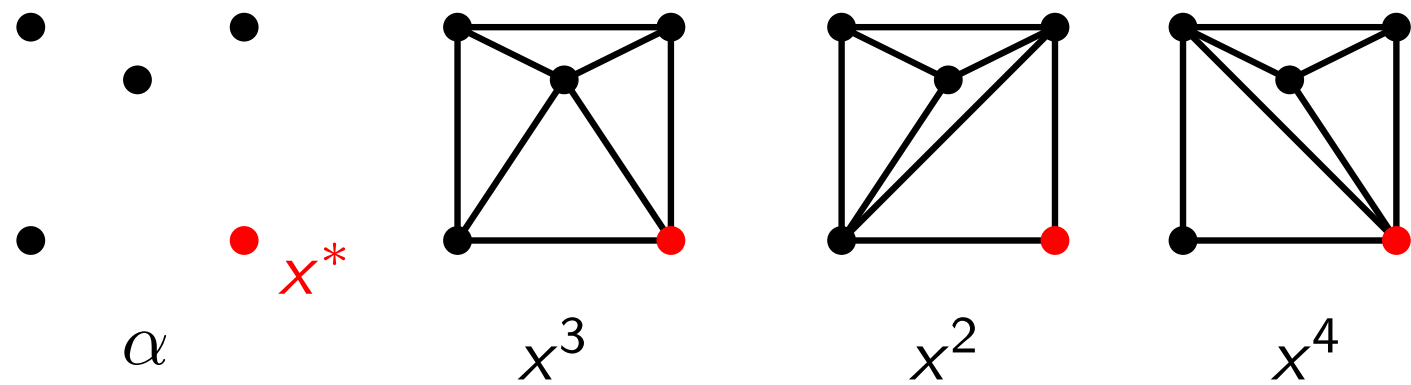
Number of triangulations?

- Same general idea with the bijection
- Involves full triangulation polynomials
 - marking degrees of proxies
 - keeping track of edges between proxies

The number of triangulations of a chirotope tree can be computed in **polynomial time** from the **precomputed** full triangulation polynomials of the nodes [BFGK]

$|\mathcal{T}_\kappa| \approx 5.92966751 \cdot 10^{180}$ computed **exactly** in a few seconds using Sagemath.

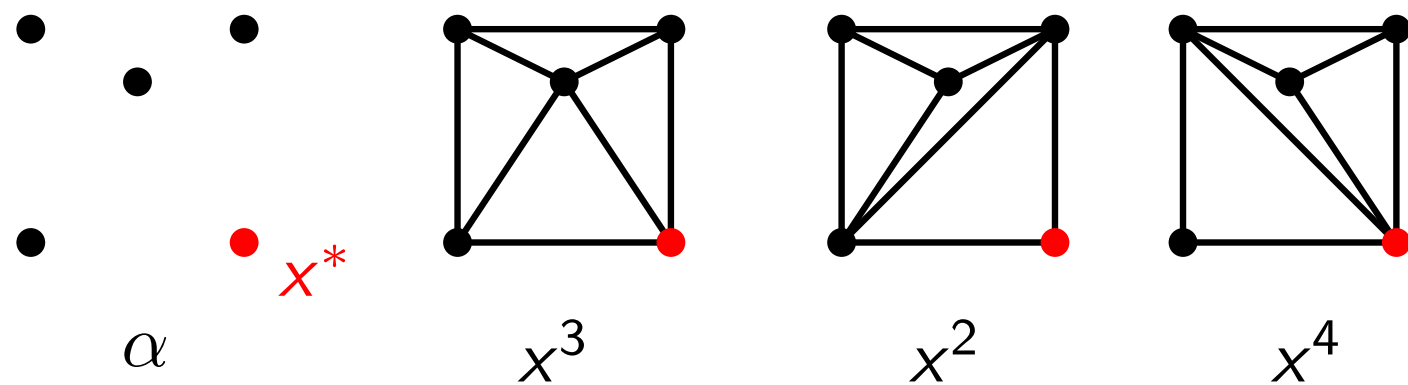
Computing triangulation polynomials



$$P_{\alpha, x^*}(x) = x^2(1 + x + x^2) \quad P_{\alpha, x^*}(1) = 3$$

First idea: enumerate all triangulations with Sage and deduce the polynomial

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→ bug in sage !

```
python3.10 43,30 Go
```

```
[FlorentKoechlinMacAir:~ florent$ sage]
SageMath version 9.6, Release Date: 2022-05-15
Using Python 3.10.3. Type "help()" for help.

sage: PointConfiguration.set_engine('internal');
....: points = [[64374, 1170], [28595, 16], [1162, 658], [28874, 3308], [29974, 943
....: 6],
....: [30590, 22299], [49434, 11393], [56042, 11982], [42392, 33338], [33404, 64
....: 878]];
....: p = PointConfiguration(points);
....: p_fine = p.restrict_to_fine_triangulations();
....: list_triangulations = list(p_fine.triangulations()) #does not seem to term
....: inate quickly
/usr/local/bin/sage: line 20: 84061 Killed: 9 /usr/bin/env - PATH="$PATH" $MIN_ENV "$SYMLINK"/venv/bin/sage "$@"
```

sagemath / sage Public

<> Code Issues 4.9k Pull requests 316 Actions

New issue Jump to bottom

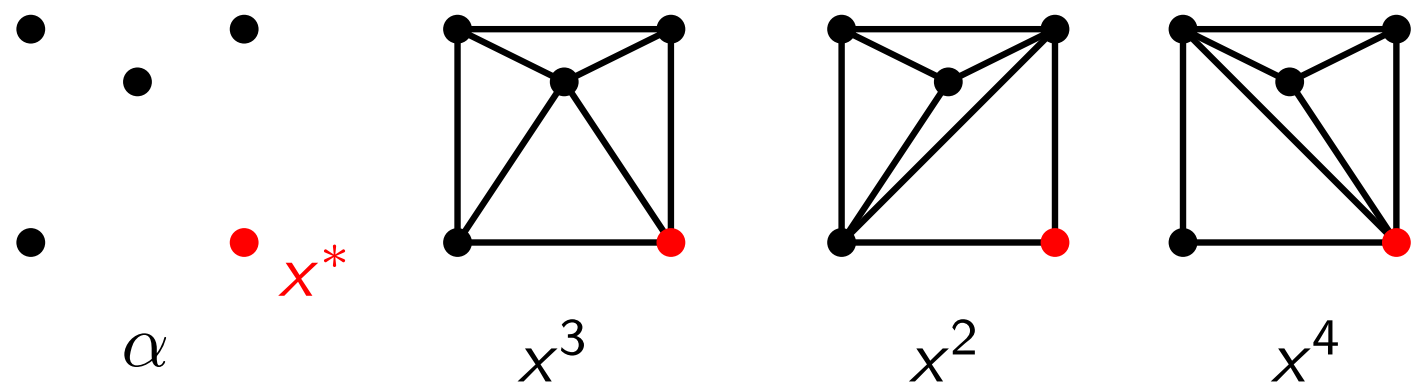
Enumerating the 2002 triangulations of a 10 point configuration kills sagemath #38187

Open 2 tasks done

seblabbe opened this issue 2 weeks ago · 2 comments

Labels: c: geometry t: bug

Computing triangulation polynomials



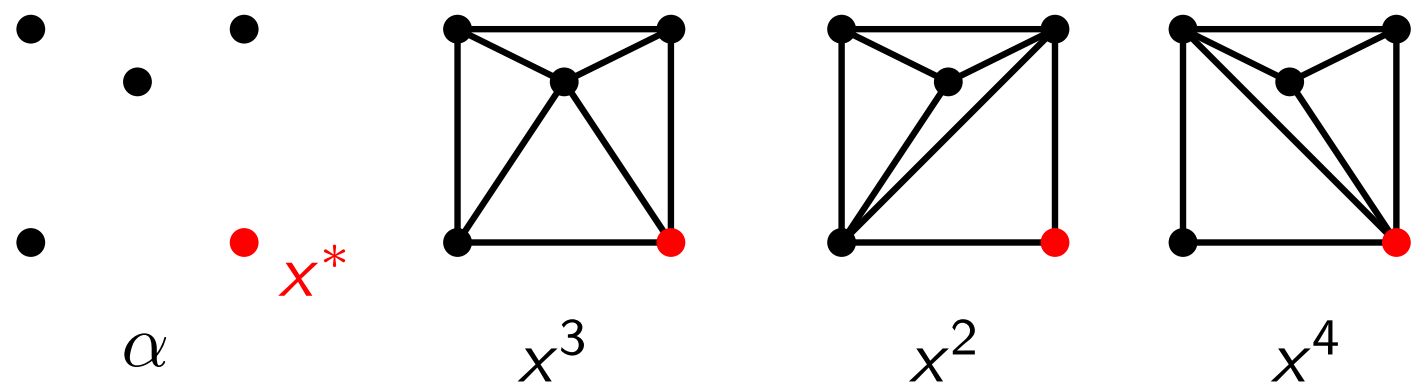
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Computing triangulation polynomials



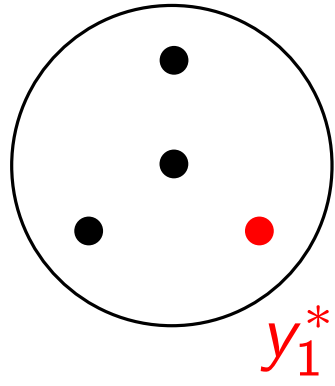
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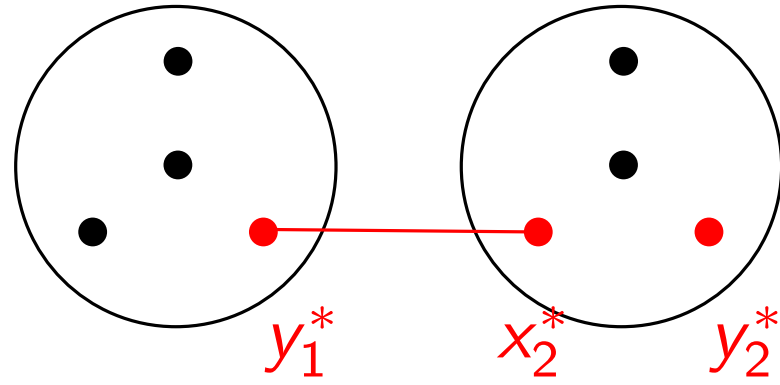
Better idea: the $\mathcal{O}(n^2 2^n)$ algorithm of [Alvarez and Seidel 2013] can be easily adapted to compute the full triangulation polynomials

Limitation: only for a fixed **given** tree. Does not apply for recursive families of chirotope trees.

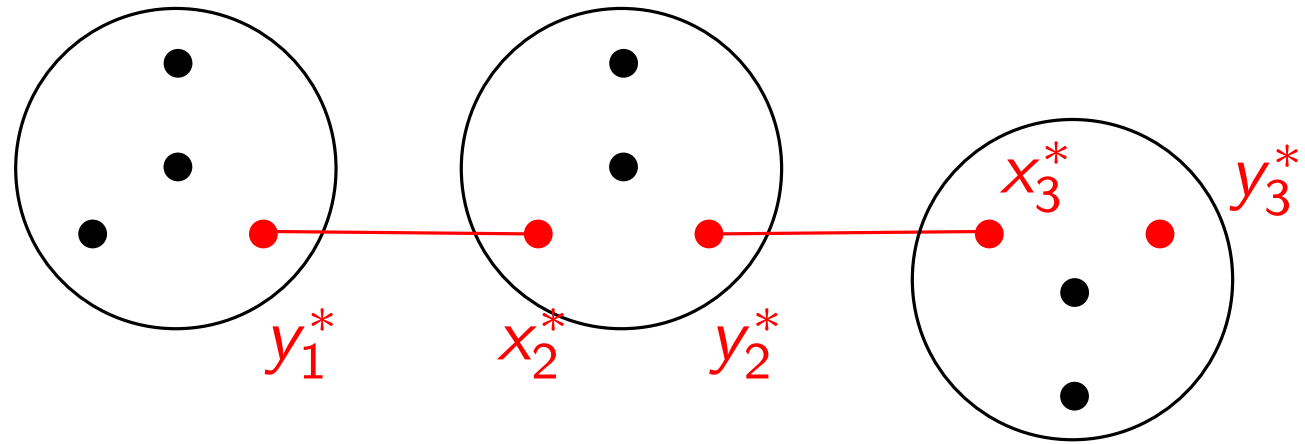
An application from analytic combinatorics



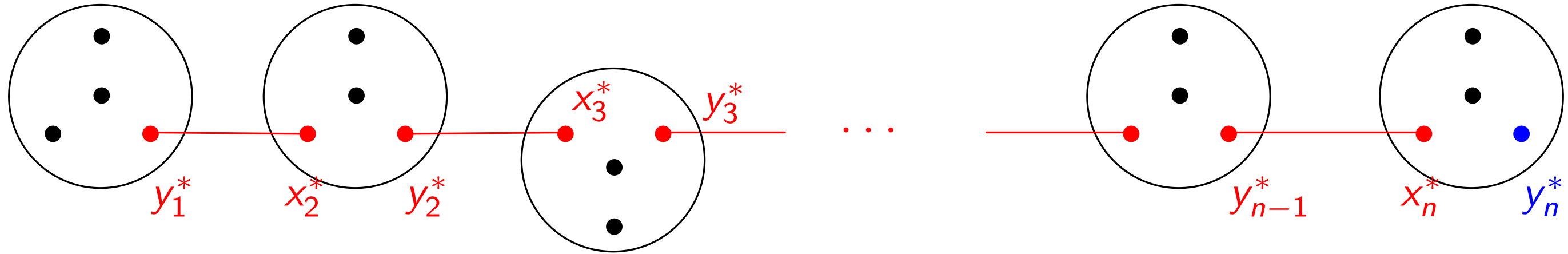
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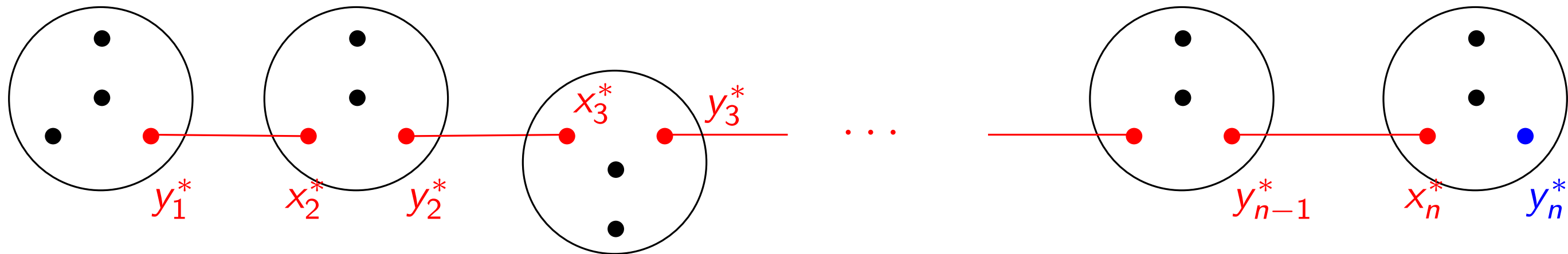


An application from analytic combinatorics



$$\alpha_{n+1} = \alpha_n y_n^* \bowtie_{x_{n+1}^*} \alpha_1$$

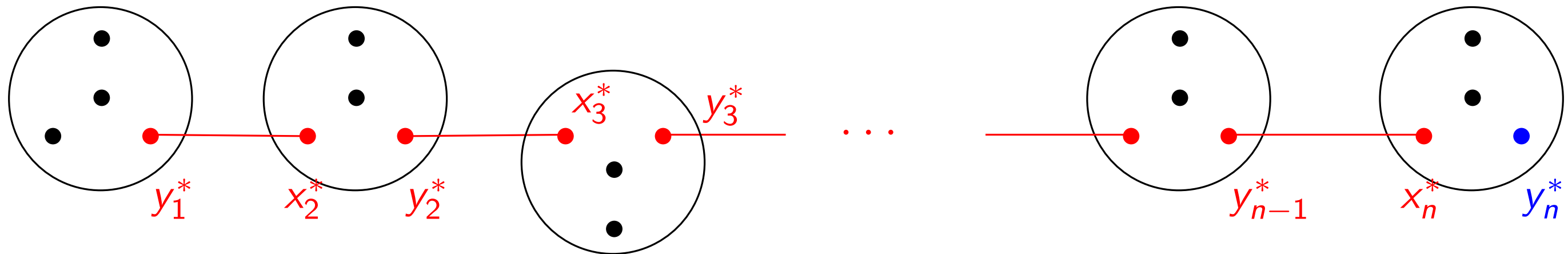
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We introduce $P_n(u) \stackrel{\text{def}}{=} P_{\alpha_n, y_n^*}(u) = \sum_{T \in \mathcal{T}_{\alpha_n}} u^{\deg_T(y_n^*)}$ and $Q_{\alpha_1, x_n^*, y_n^*}(s, u) = s^3 u^3$

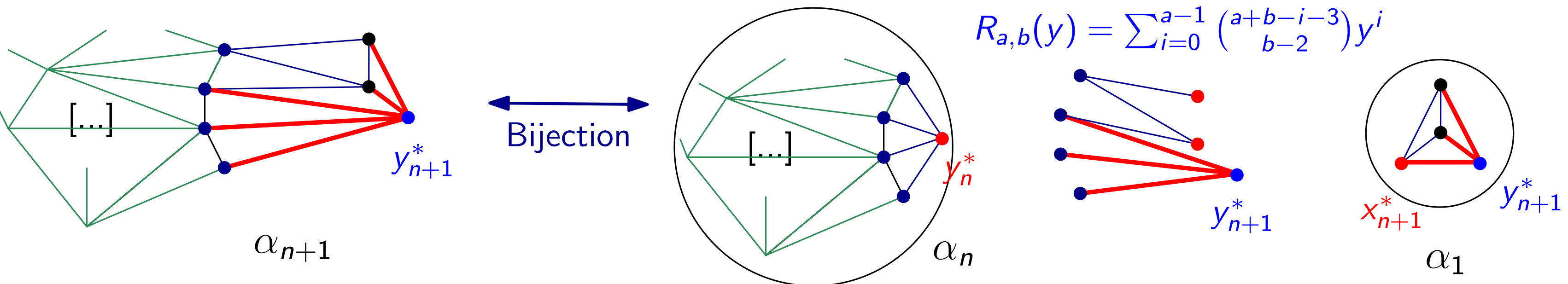
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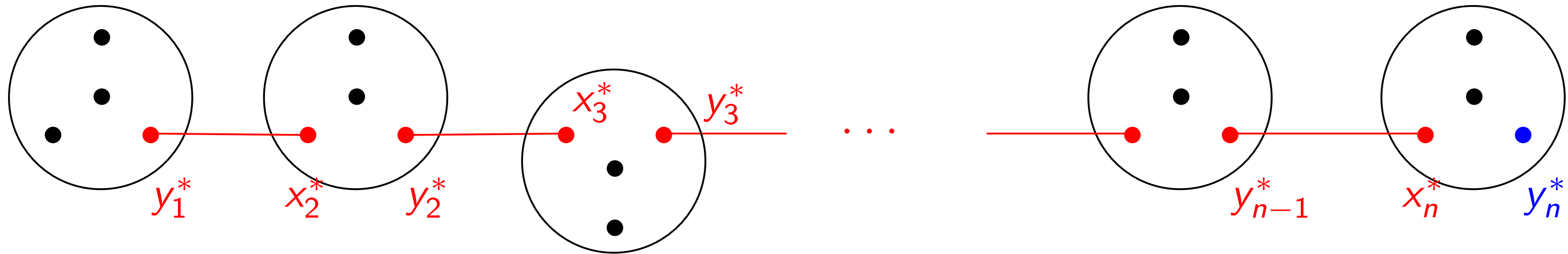
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$$P_{n+1}(u) = \sum_{a, b \geq 2} R_{a, b}(u) [u^a] P_n(u) [s^b] Q_{\alpha_1, x_n^*, y_n^*}(s, u) = \sum_{a \geq 2} [u^a] P_n(u) u^3 \sum_{i=0}^{a-1} (a-i) u^i$$



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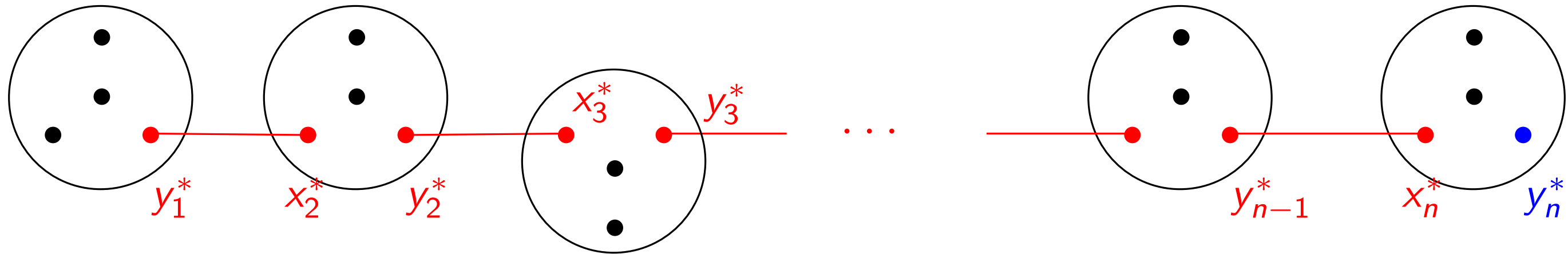


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An application from analytic combinatorics

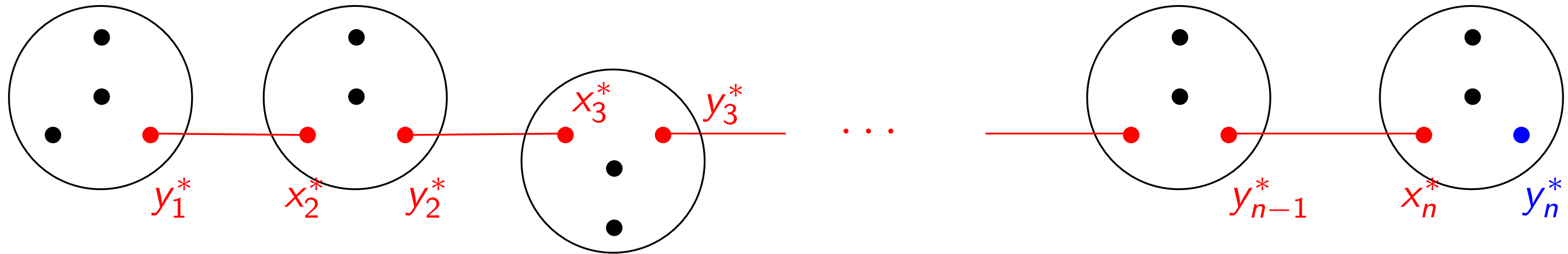


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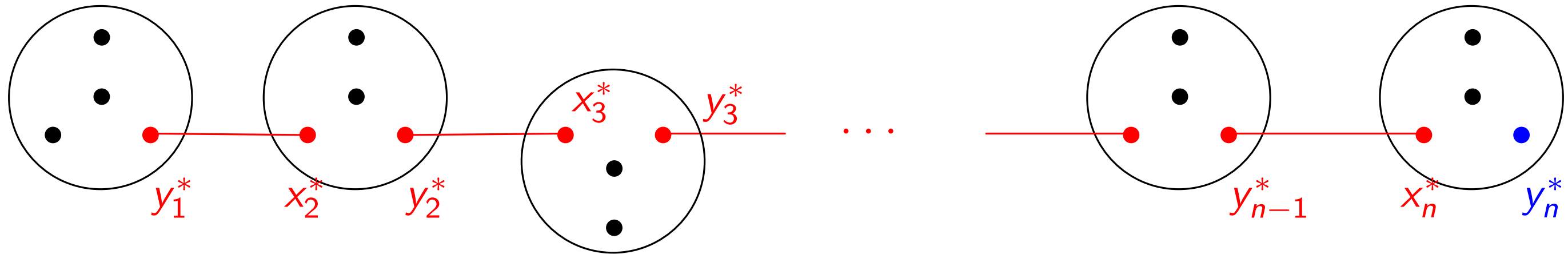
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$$\left(1 - \frac{xu^4}{(1-u)^2}\right) F(u, x) = xu^3 \left(1 - \frac{u}{(1-u)^2} F(1, x) + \frac{1}{1-u} \partial_u F(1, x)\right)$$

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Kernel method: find two $u(x)$ analytic canceling the **kernel** to reduce to a linear system

We obtain $F(1, x) = \sum_{n \geq 1} |\mathcal{T}_{\alpha_n}| x^n$ (A066357)

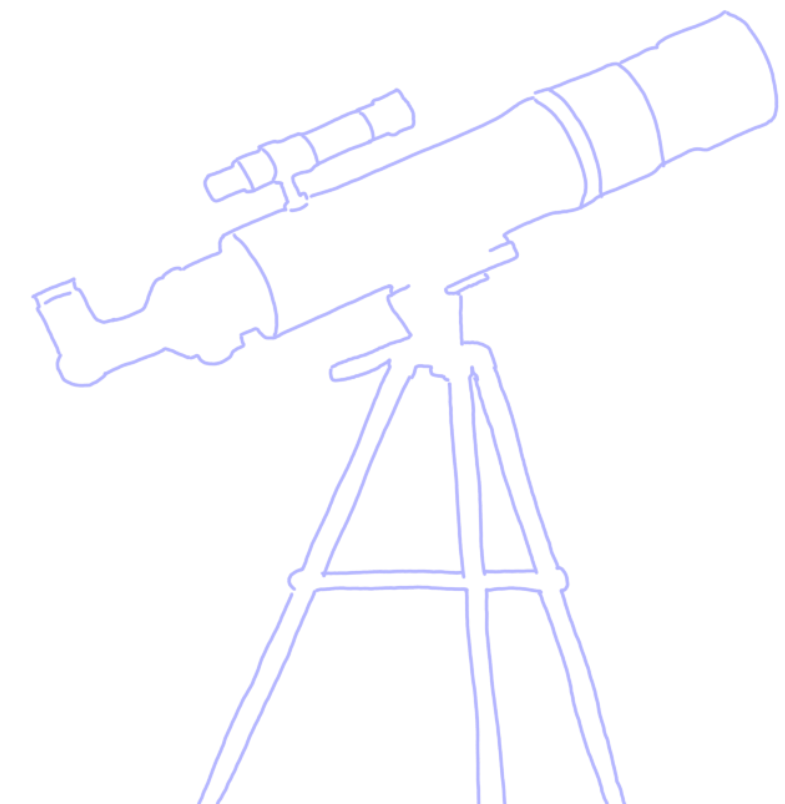
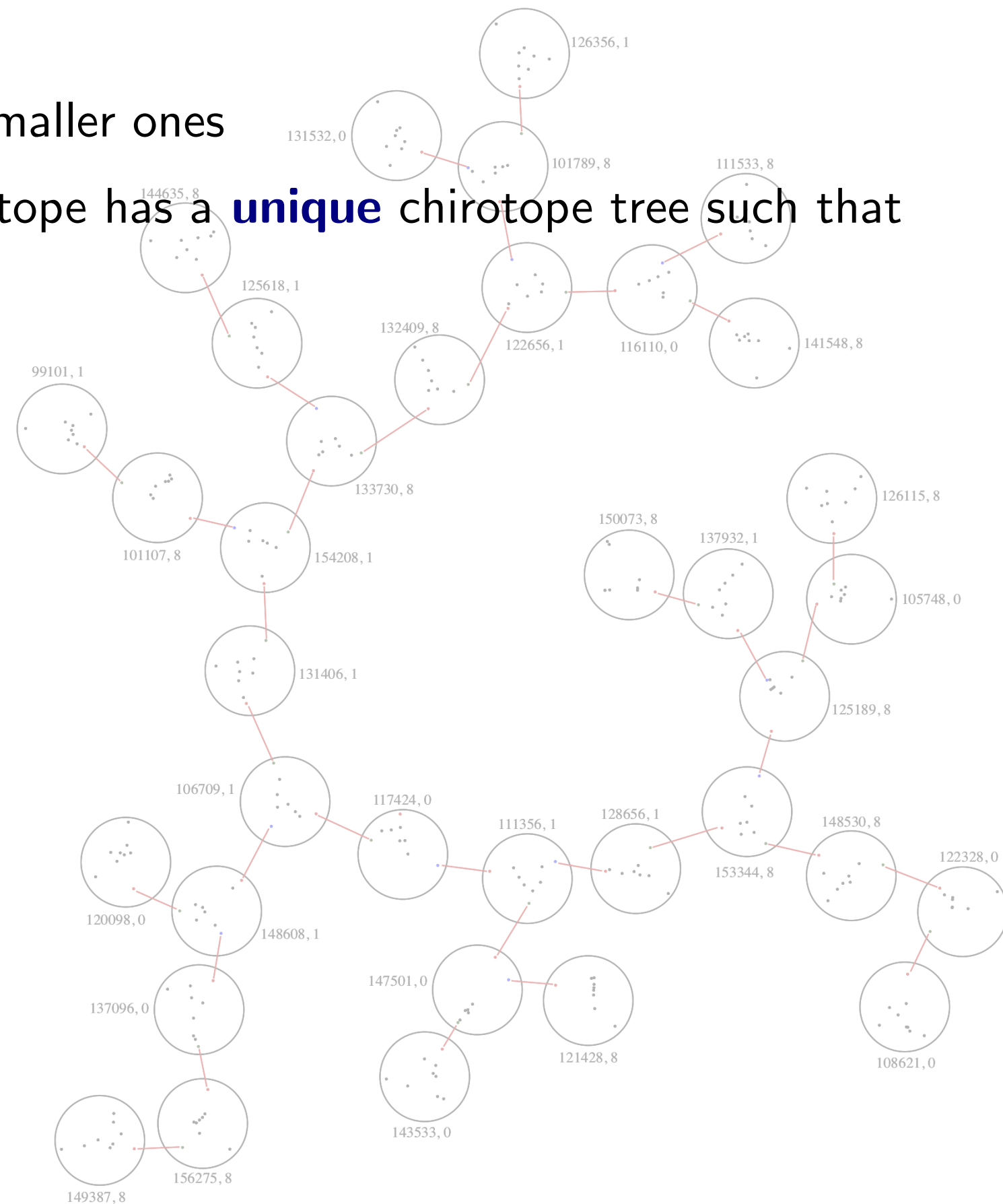
$$|\mathcal{T}_{\alpha_n}| \sim_{n \rightarrow \infty} \frac{3 - 2\sqrt{2}}{16\sqrt{2\pi}} \frac{4^{2n+2}}{n^{3/2}}$$

Conclusion

Chirotope tree : one way of building big chirotopes from smaller ones

Actually it is a **well defined** operation, every labeled chirotope has a **unique** chirotope tree such that

- each node contains an indecomposable chirotope
- there is no edge between two convex chirotopes

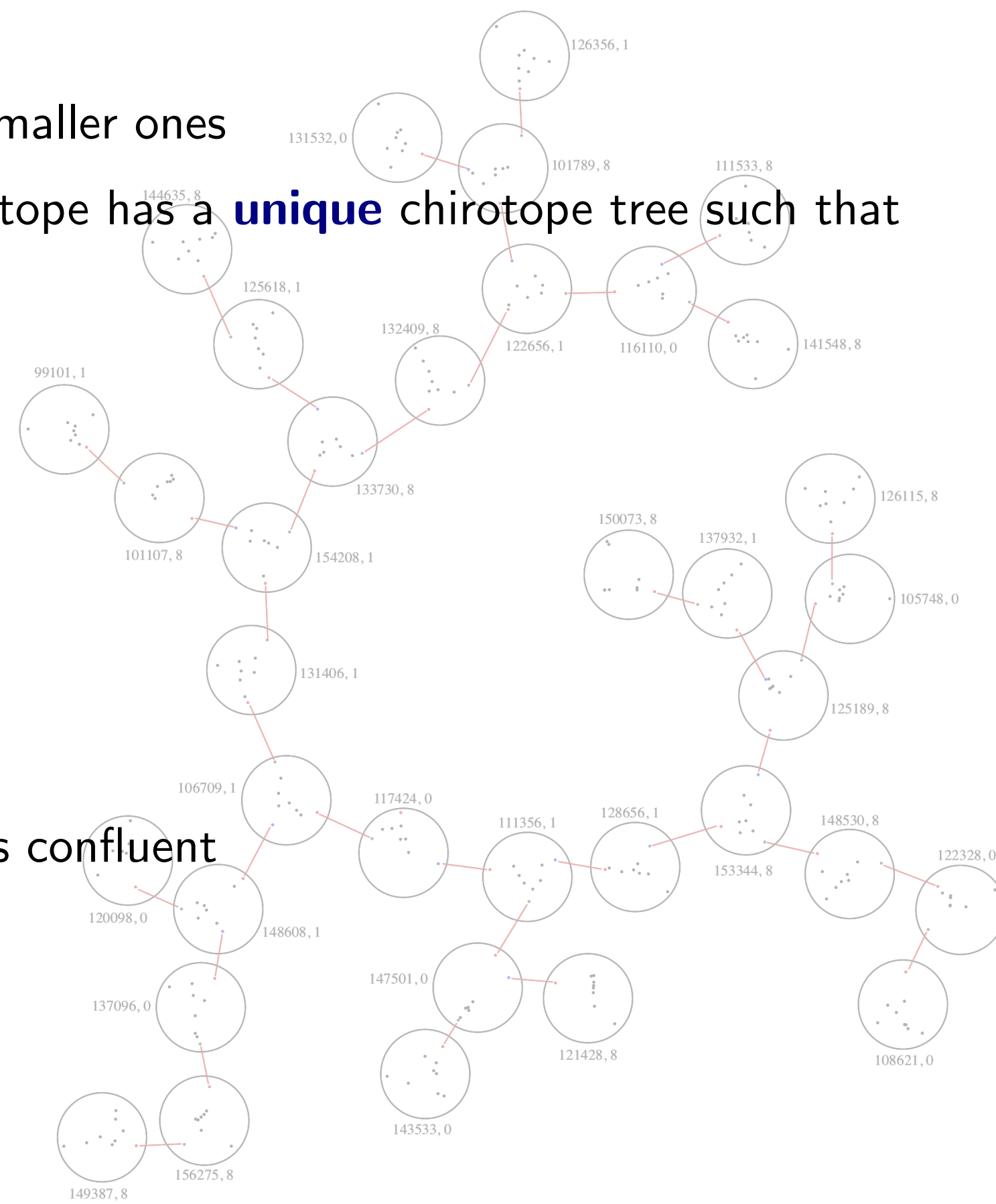
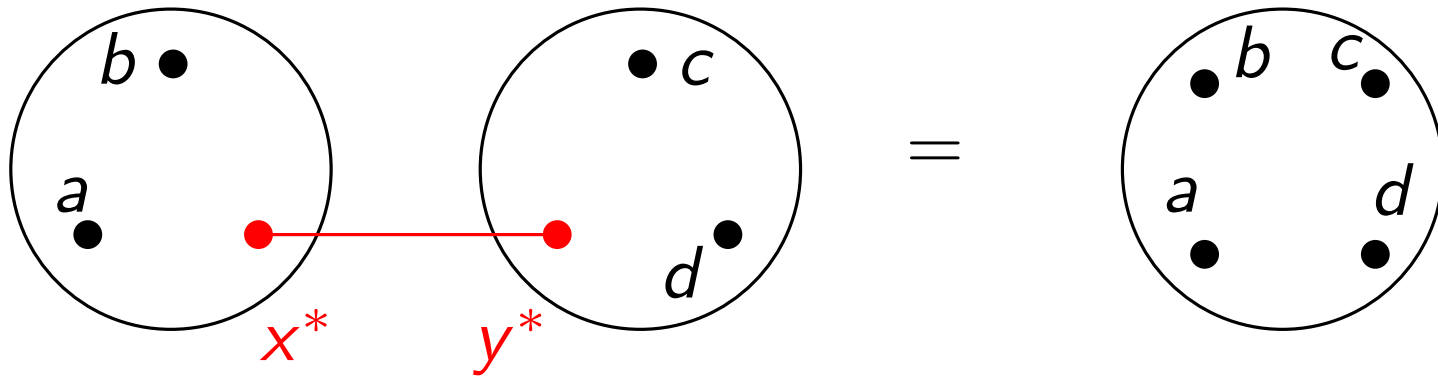


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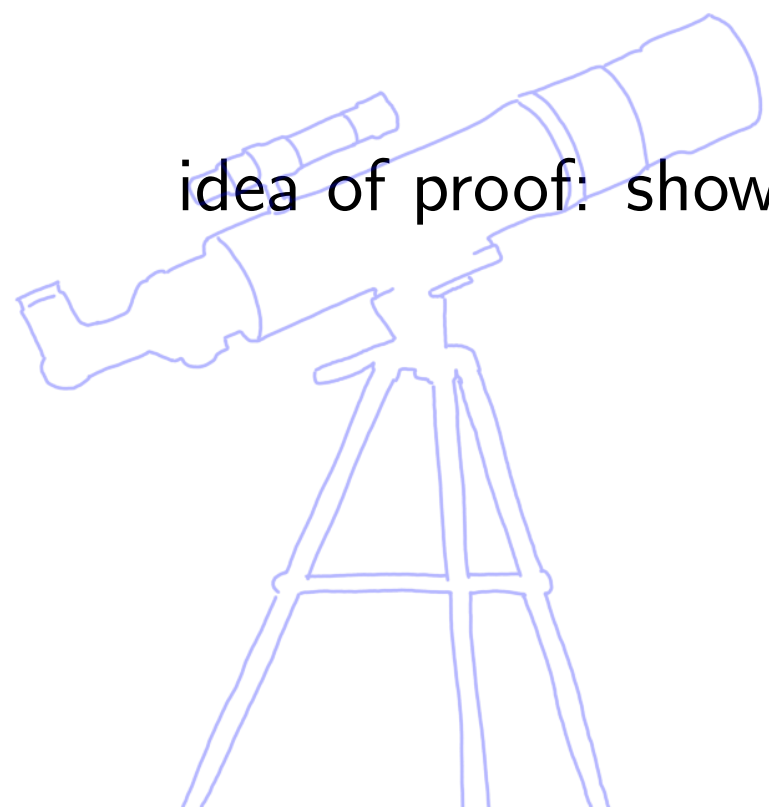
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idea of proof: showing that a split and merge process is confluent



Conclusion

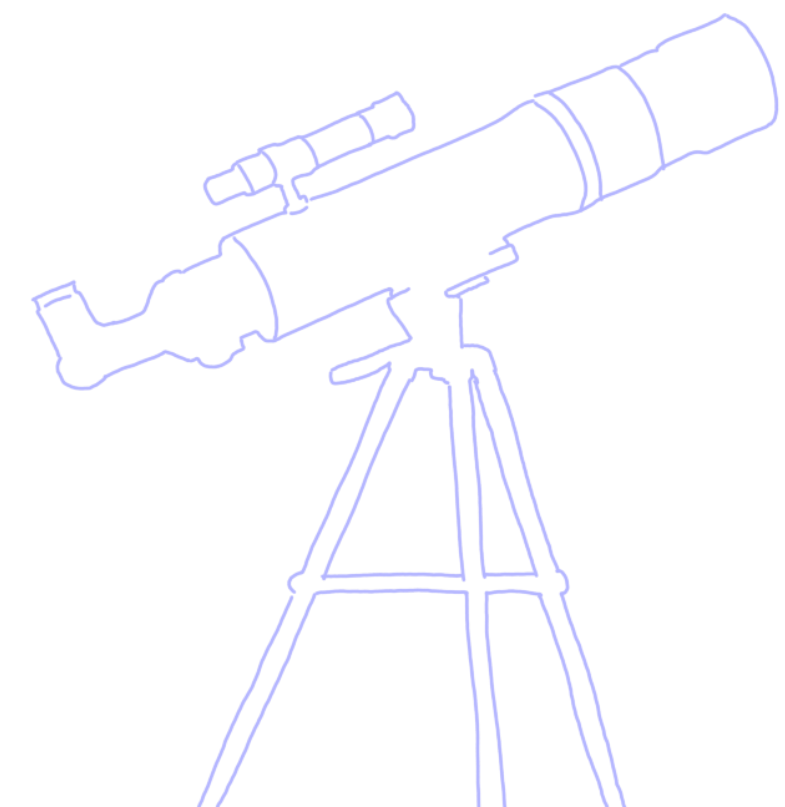
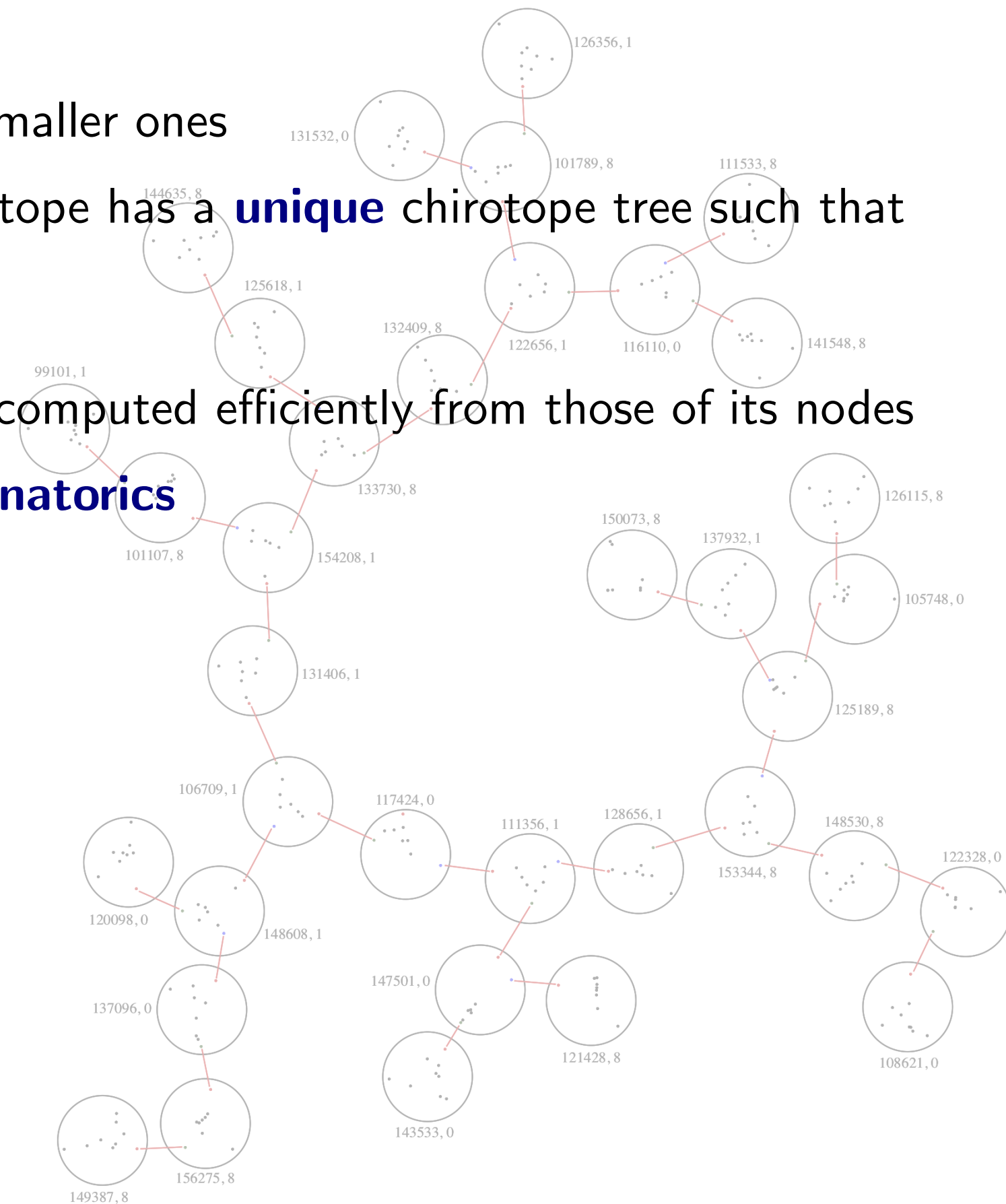
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Some chain-like trees are analyzable using **analytic combinatorics**



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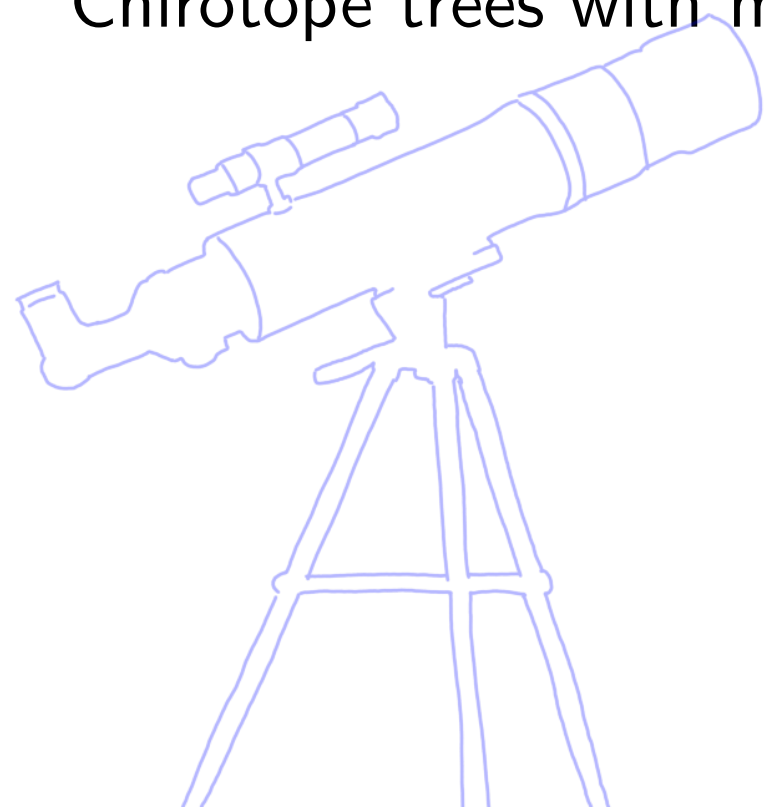
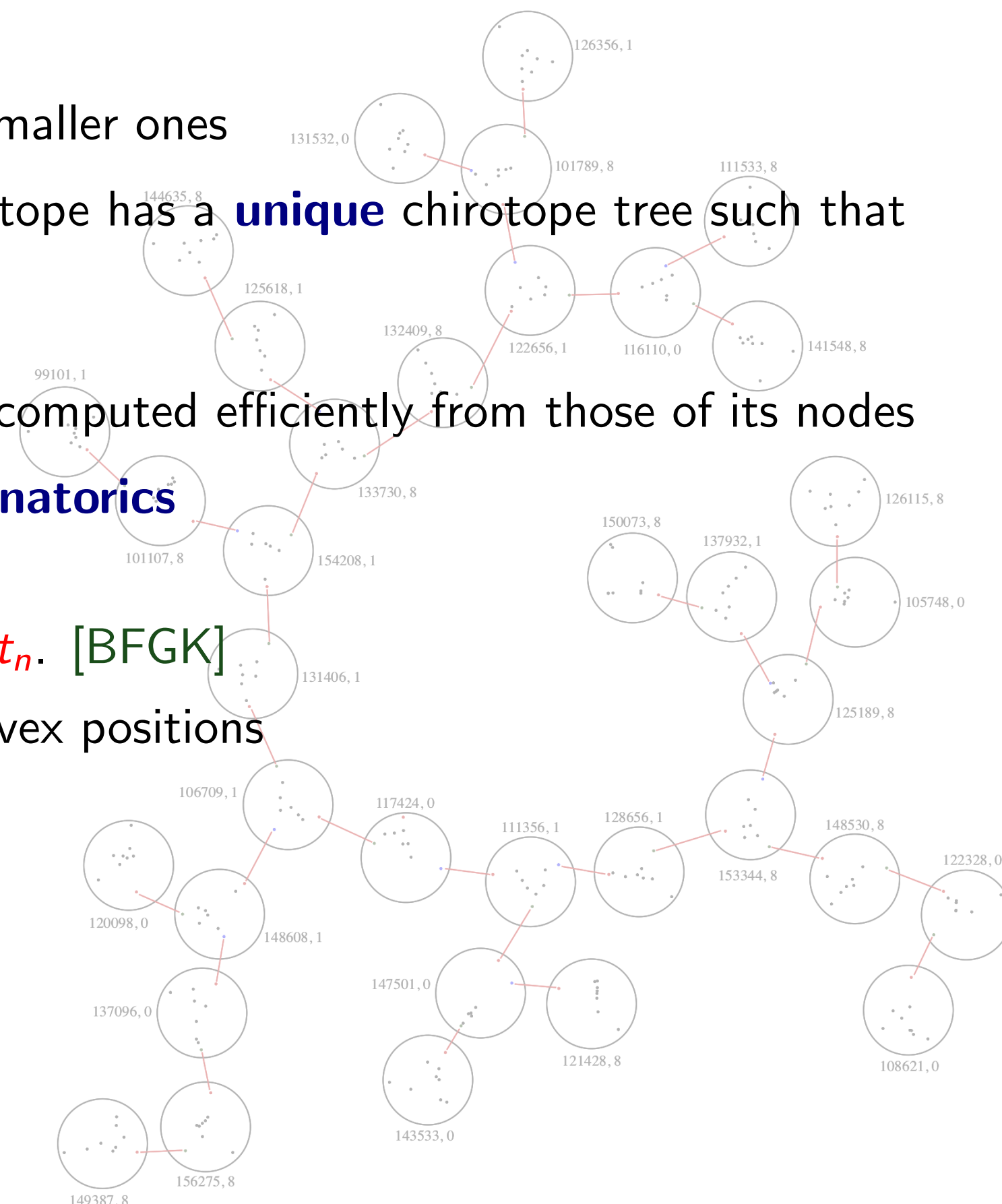
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Some chain-like trees are analyzable using **analytic combinatorics**

This is a **telescope**: $3(n - \mathcal{O}(n - 2))t_{n-1} \leq d_n \leq \mathcal{O}(n^{-3})t_n$. [BFGK]

Chirotope trees with many nodes promote elements in convex positions



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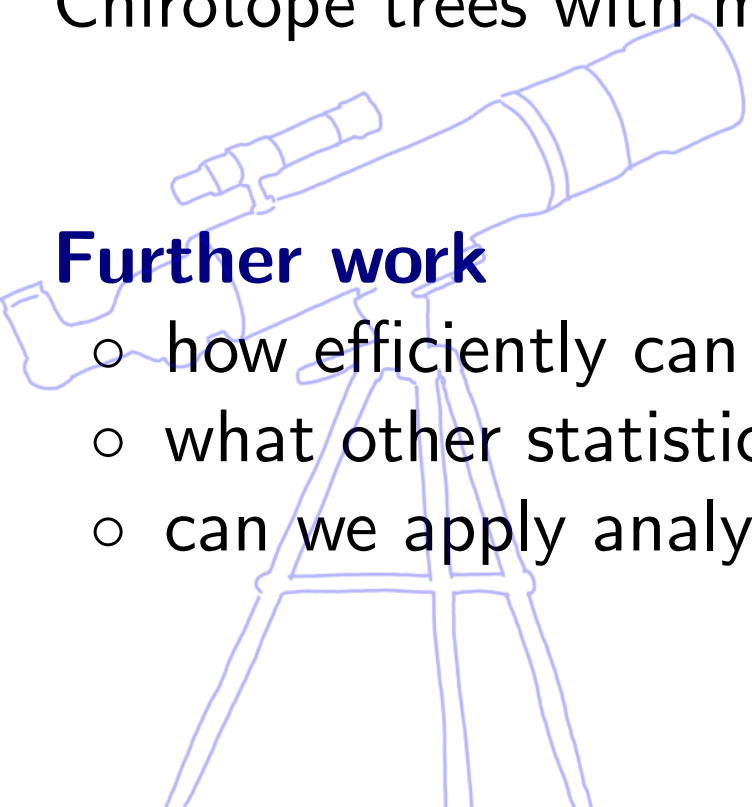
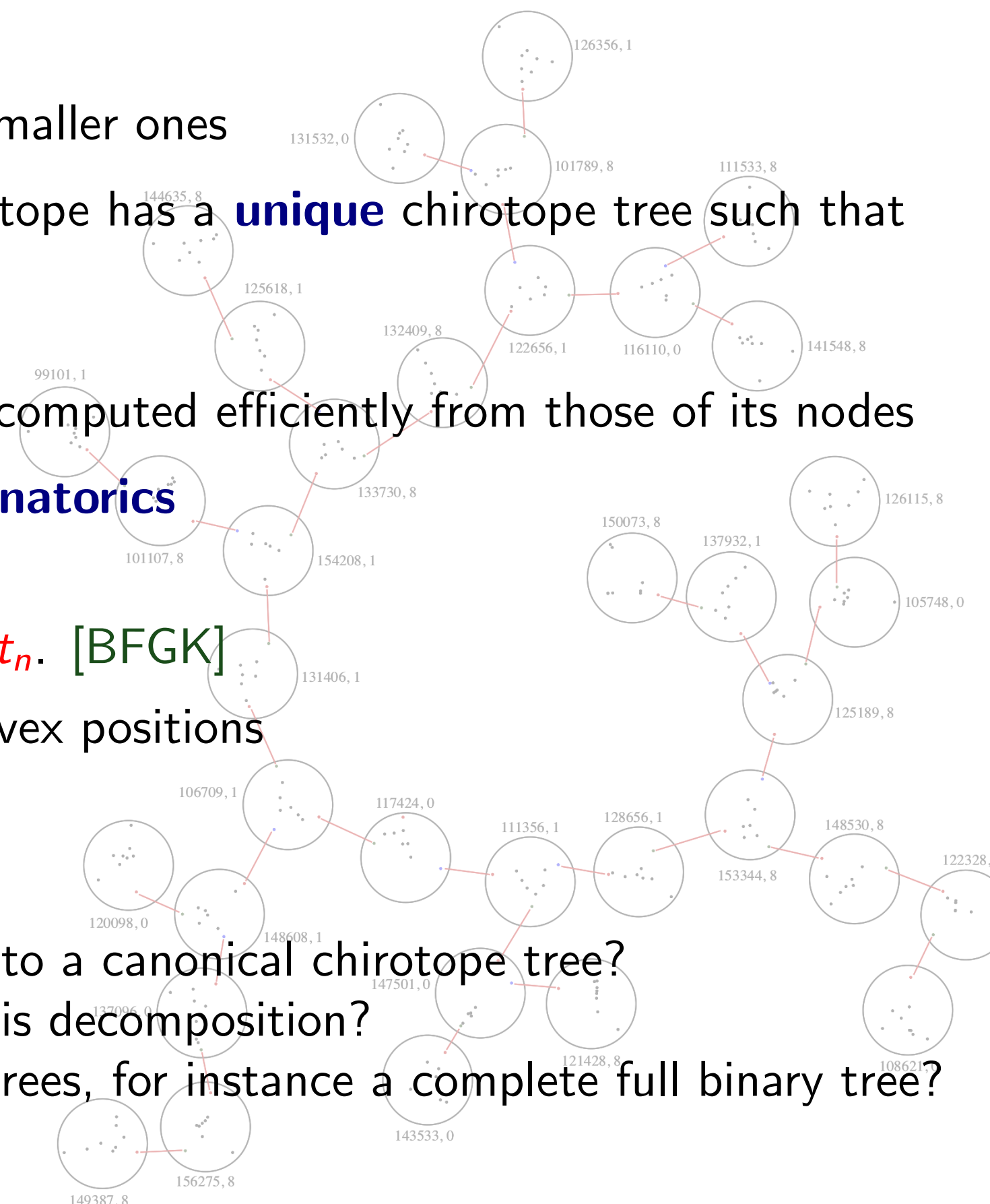
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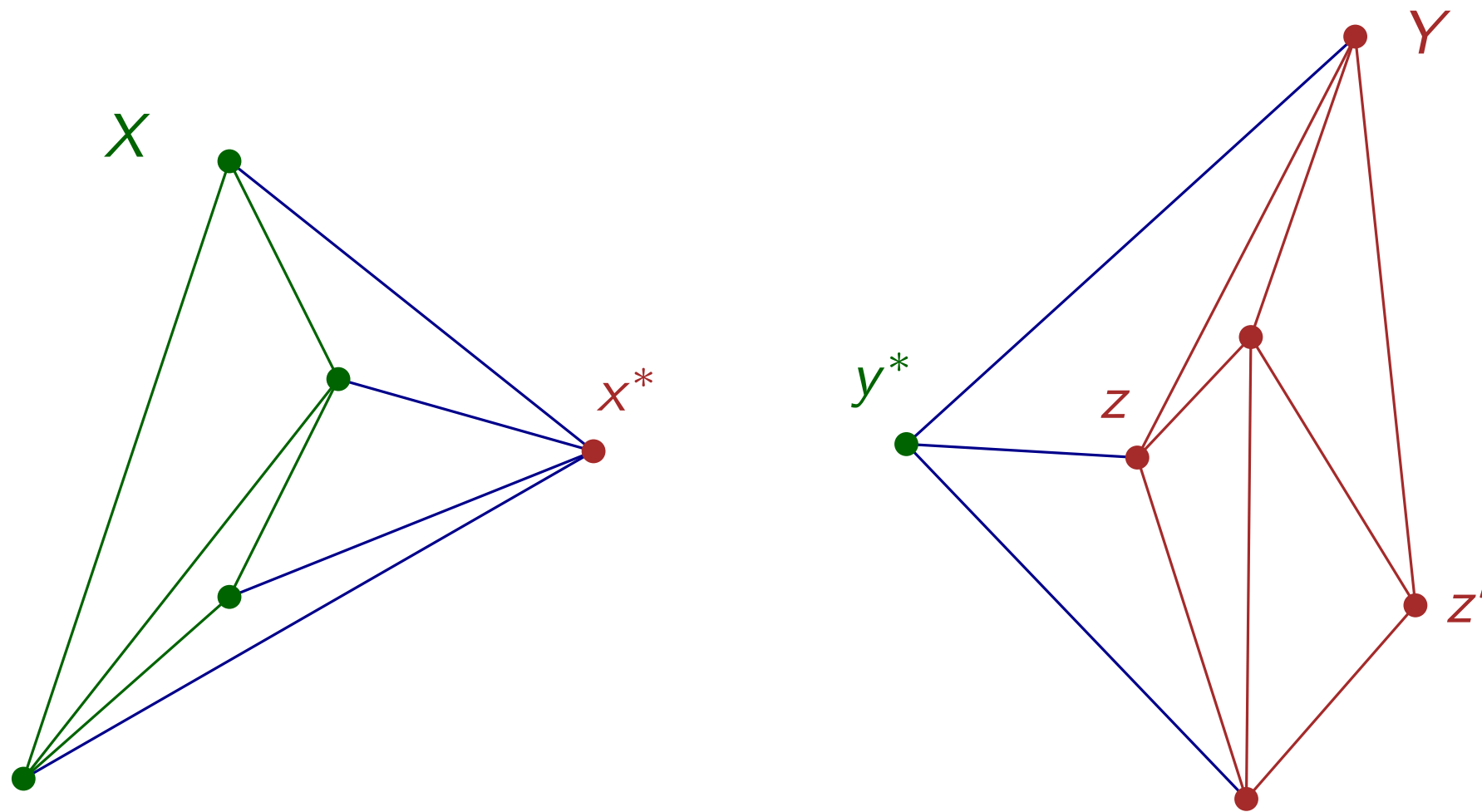
Chirotope trees with many nodes promote elements in convex positions

Further work

- how efficiently can one decompose a given chirotope into a canonical chirotope tree?
- what other statistics can be computed efficiently via this decomposition?
- can we apply analytic combinatorics to more complex trees, for instance a complete full binary tree?

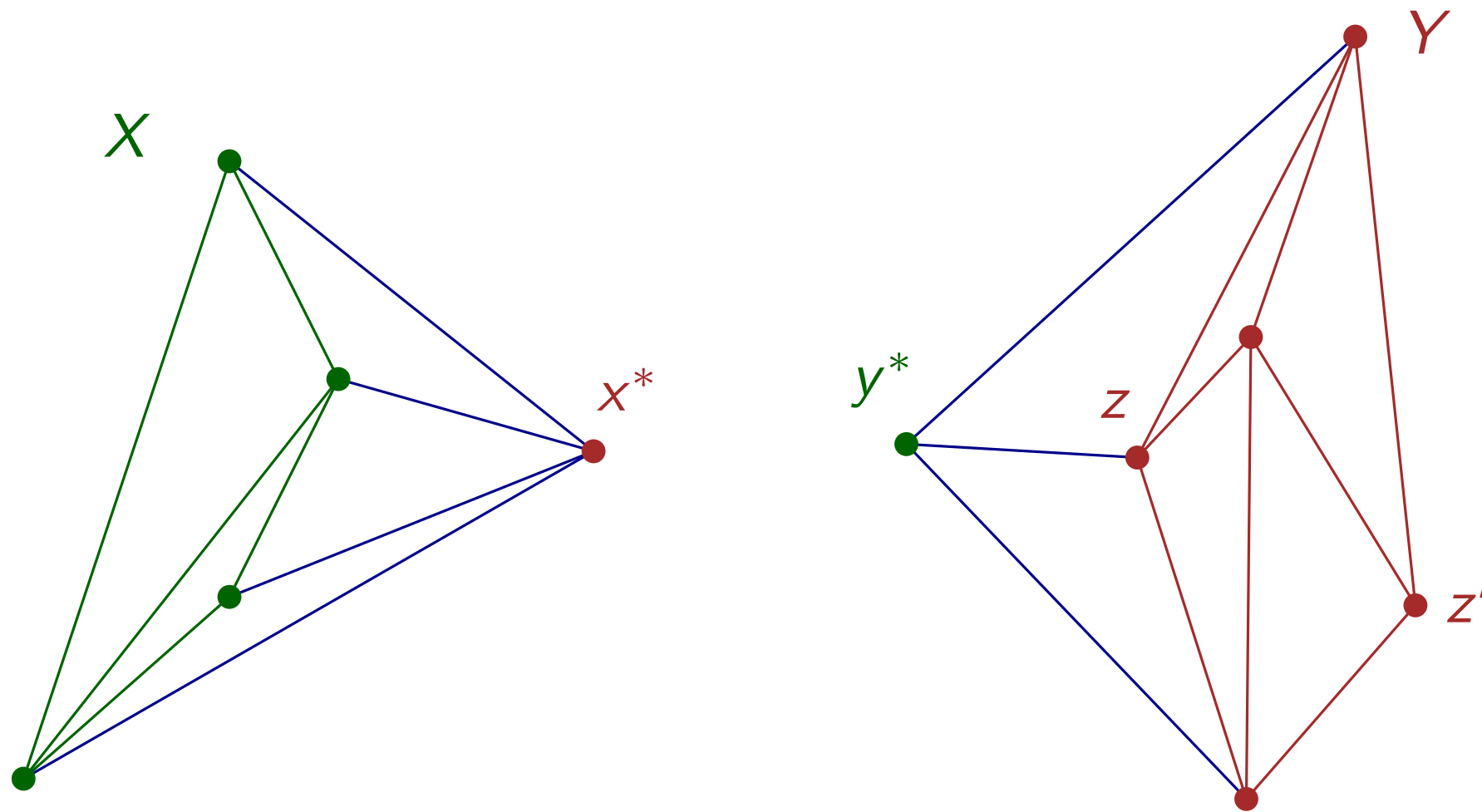


Further work: faster computation



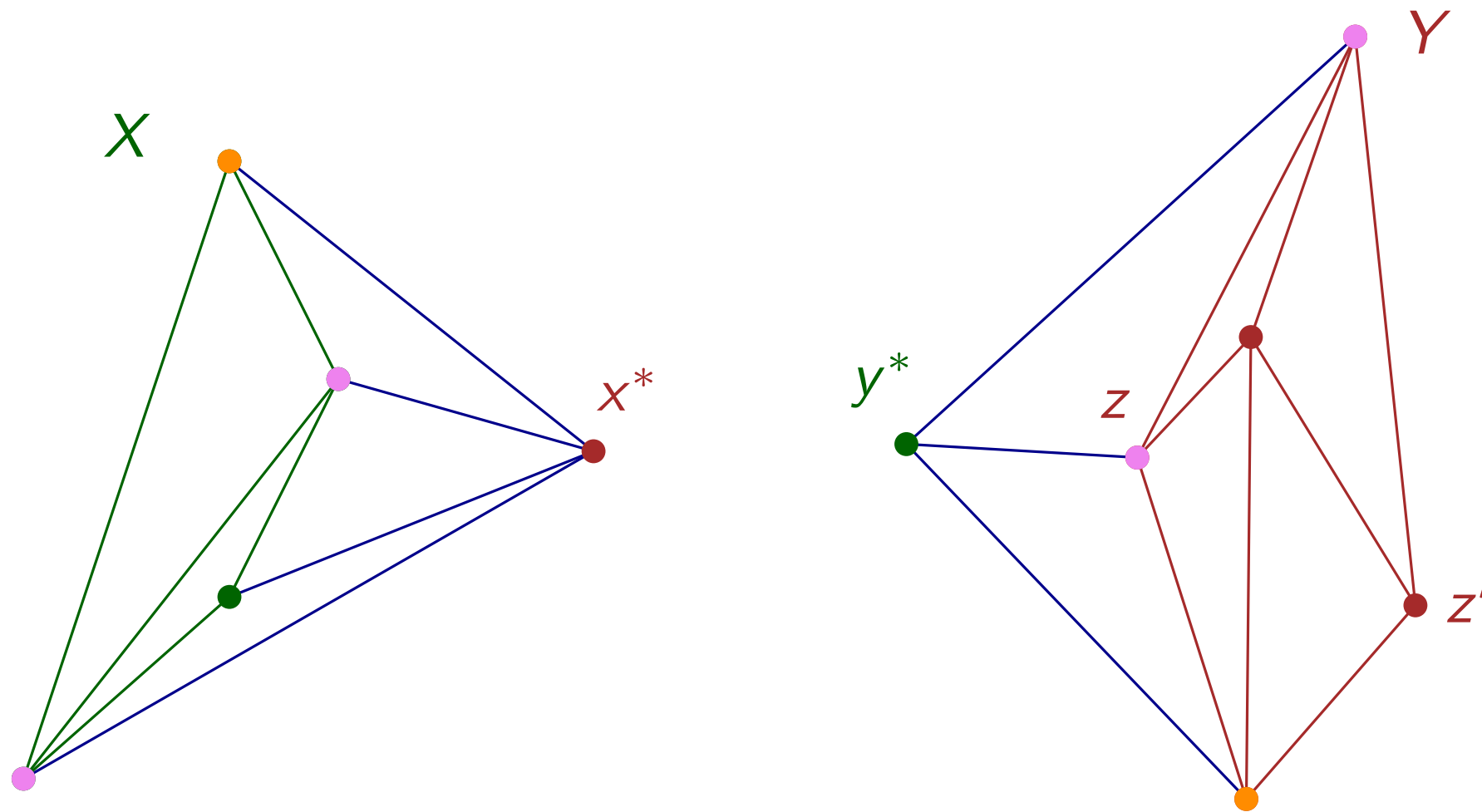
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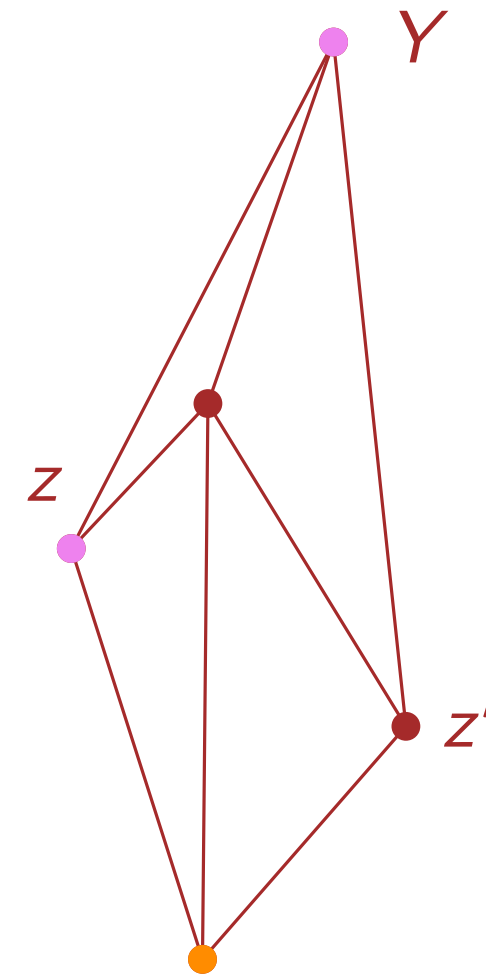
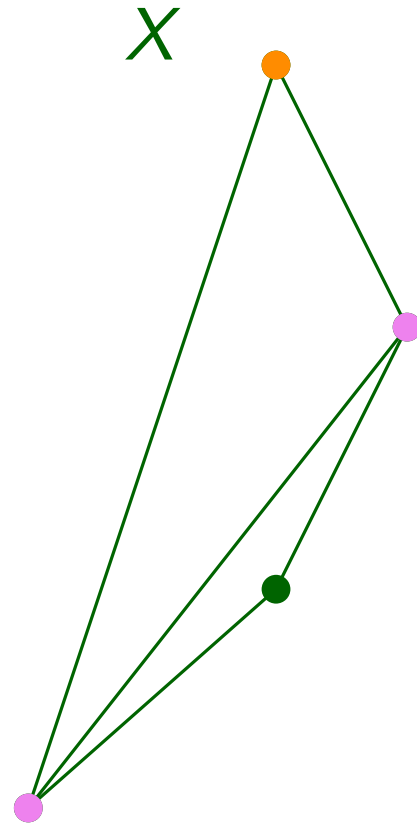
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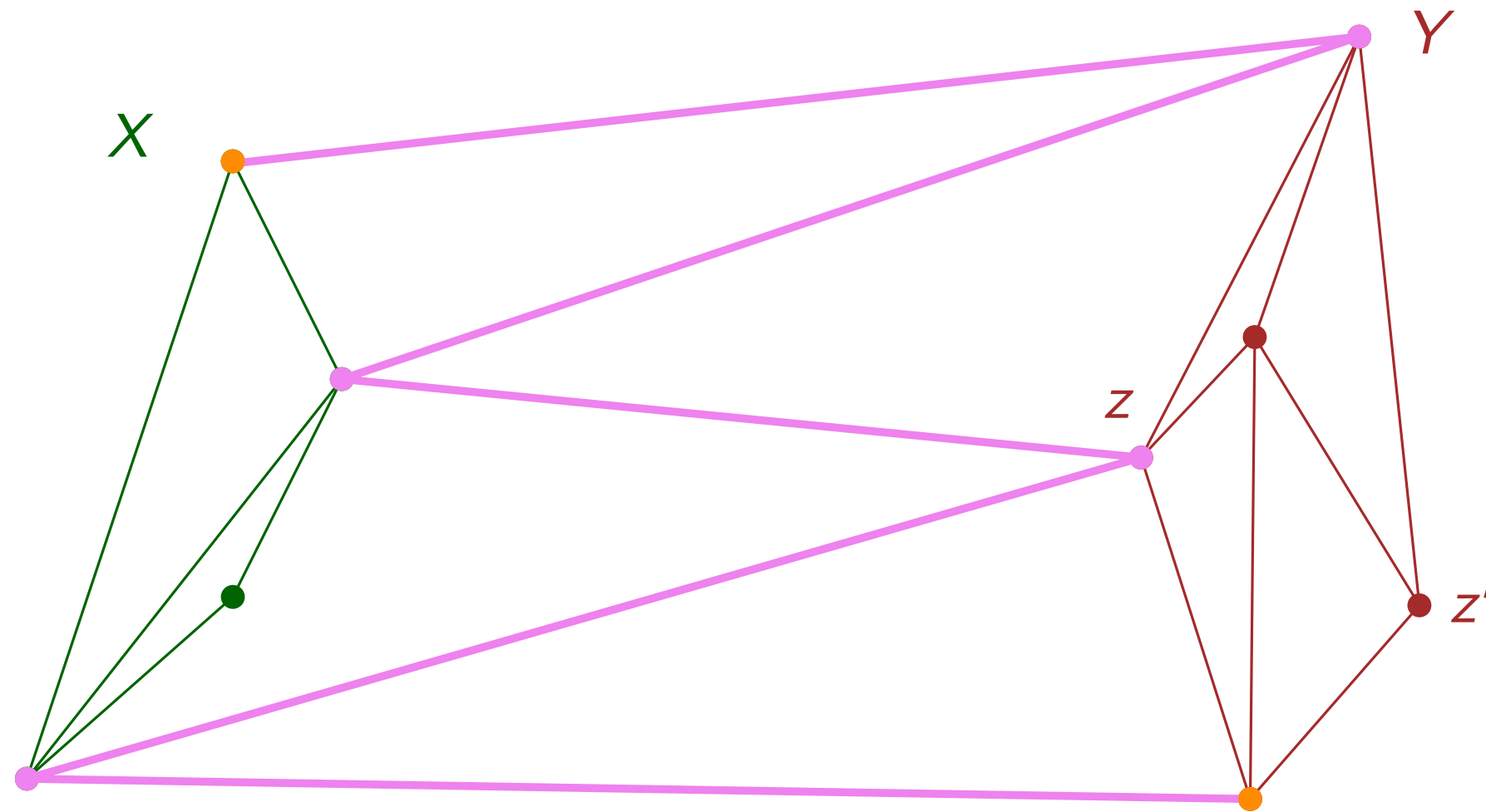
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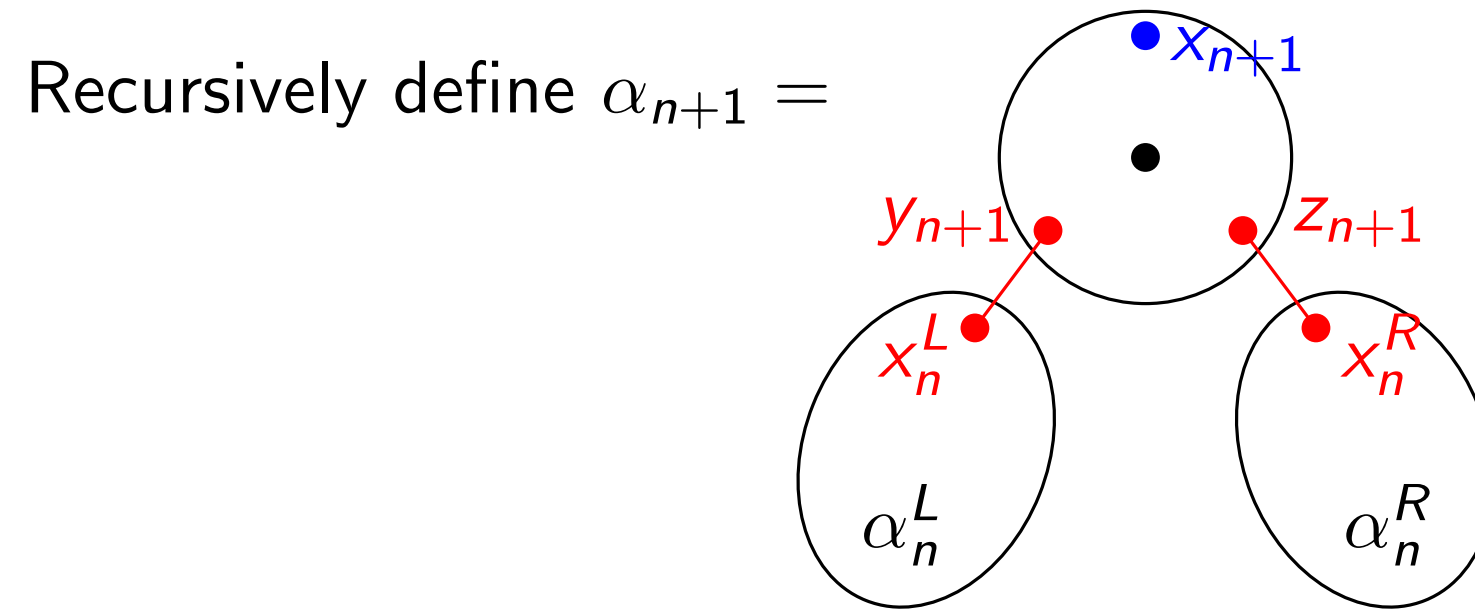
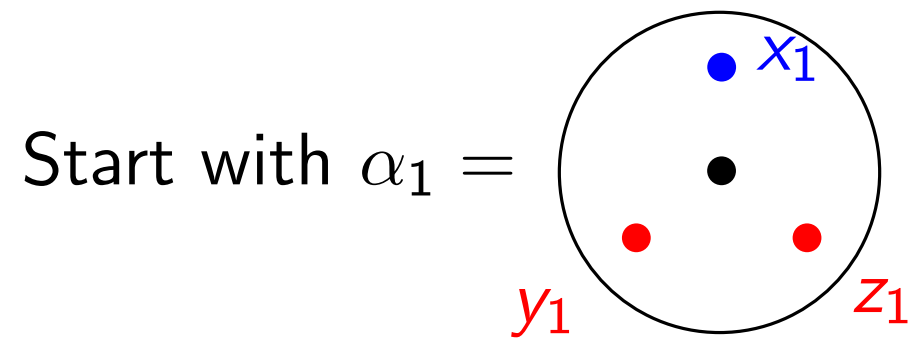
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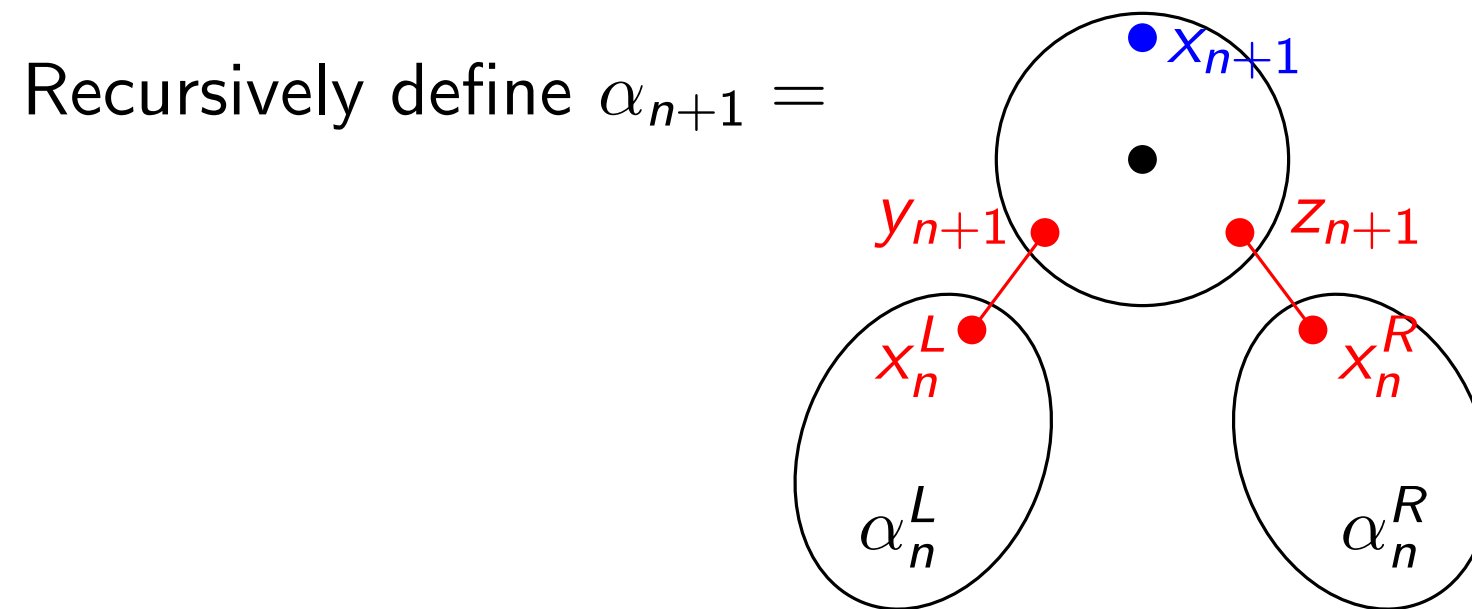
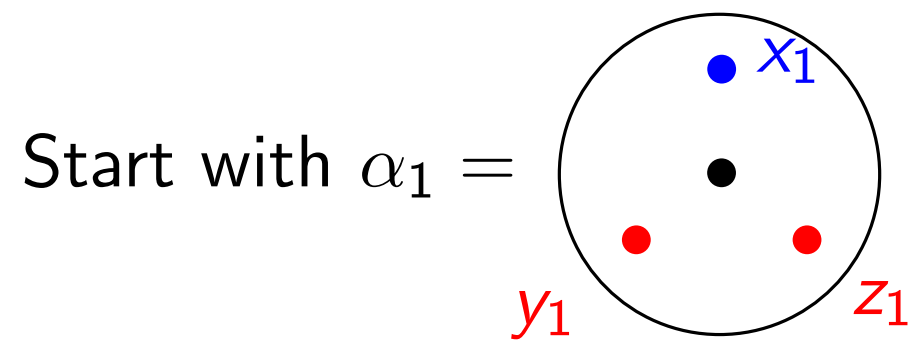


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Further work: Full binary chain



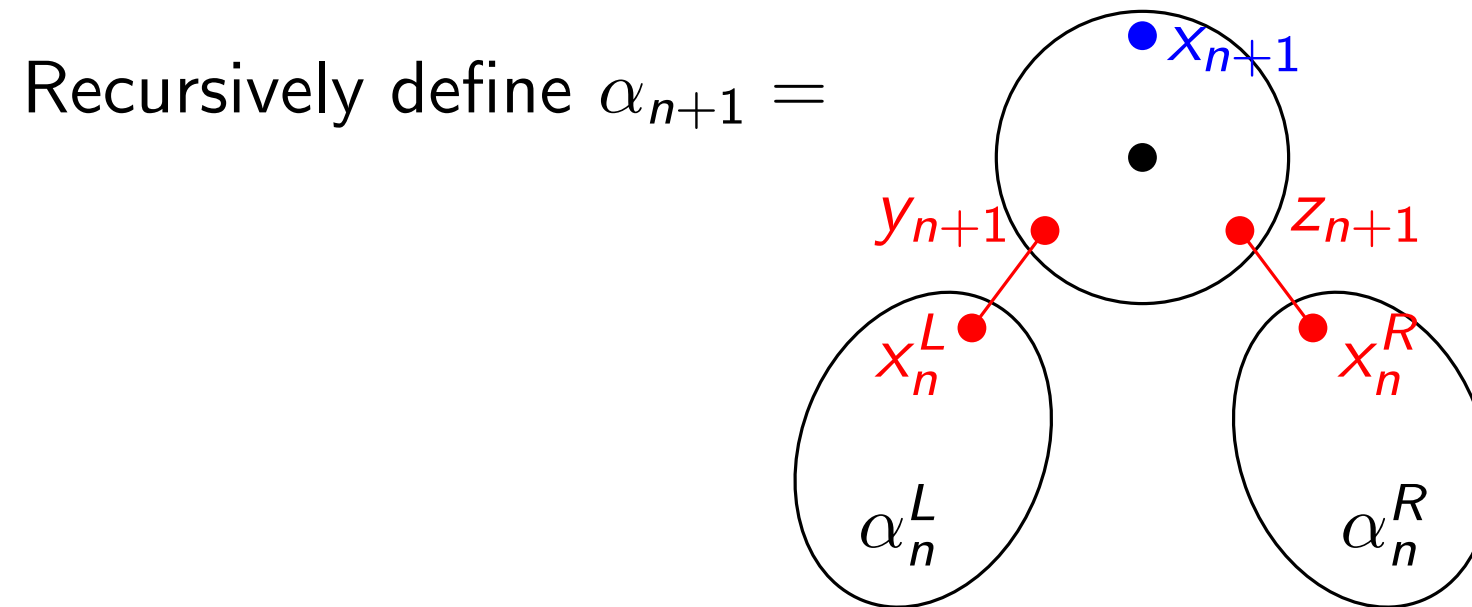
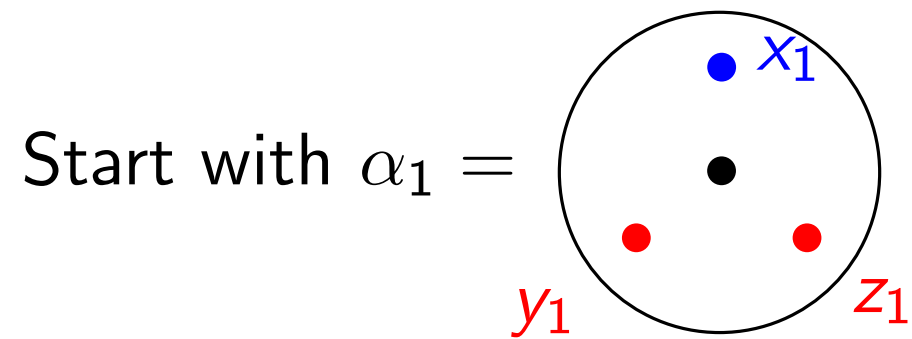
Further work: Full binary chain



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$$Q_{n+1}(x) = (x + 1)^4 \sum_{k \geq 2} \left(\Delta^{(k)}(Q_n(x)) \right)^2$$

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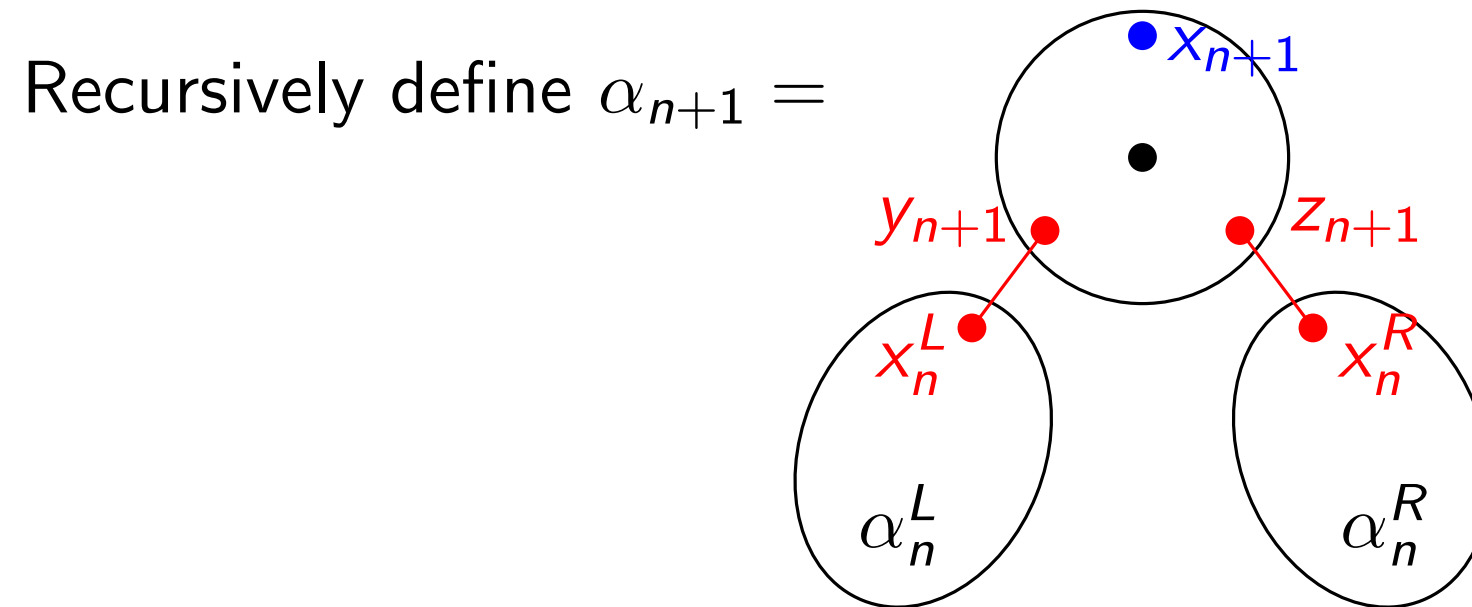
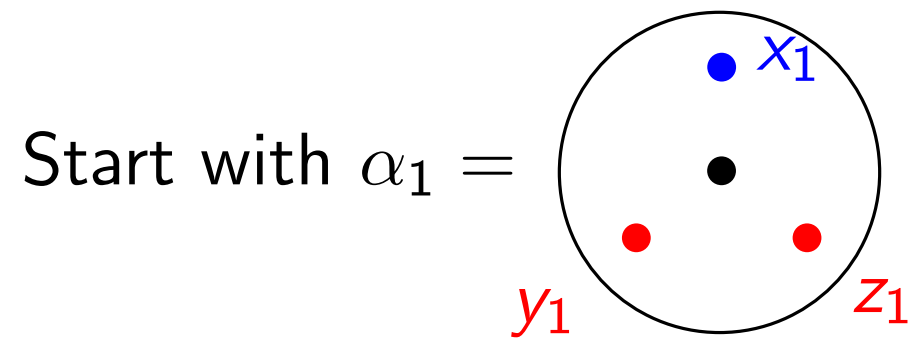


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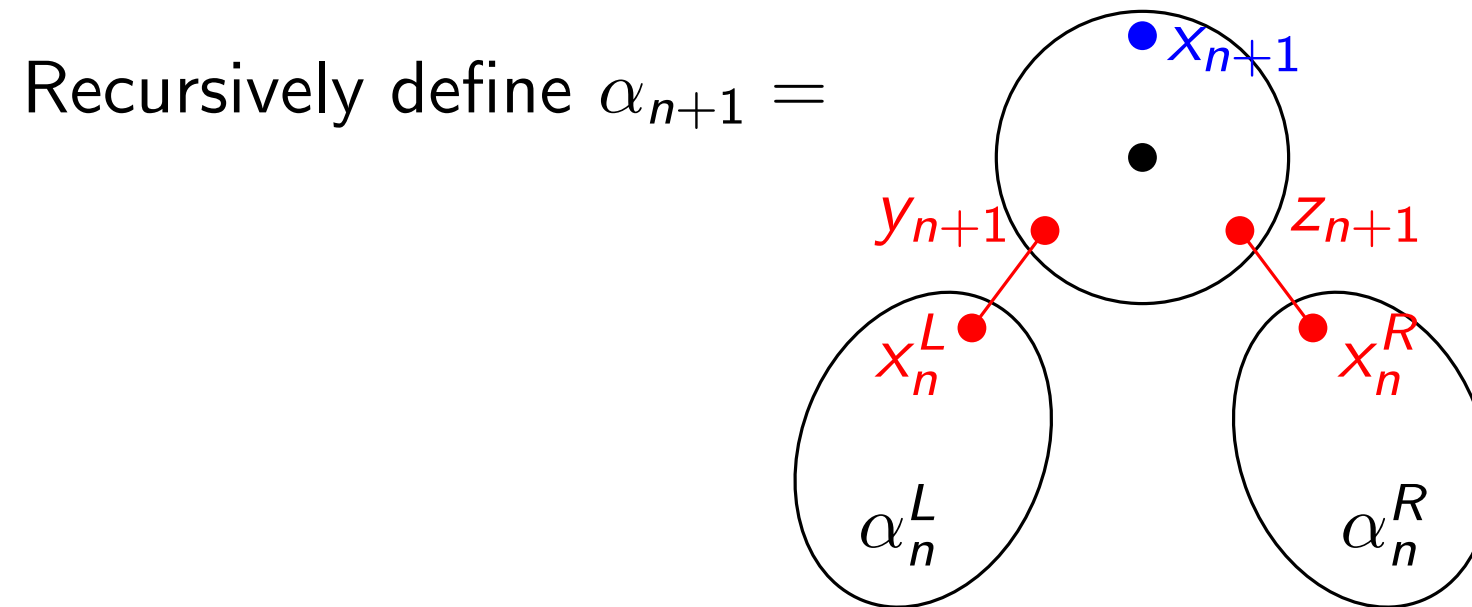
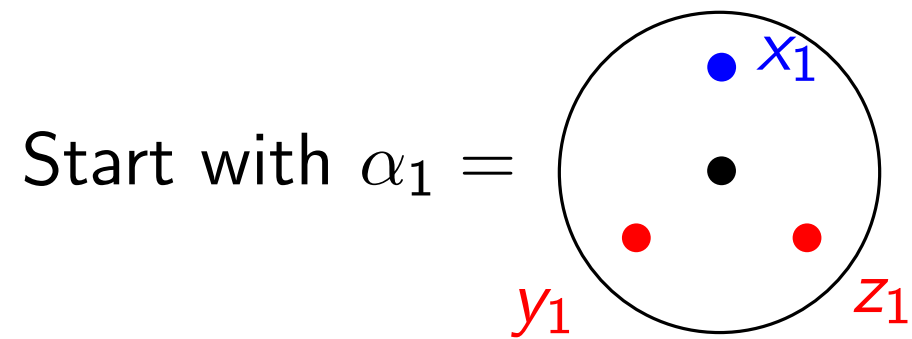
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Further work: Full binary chain



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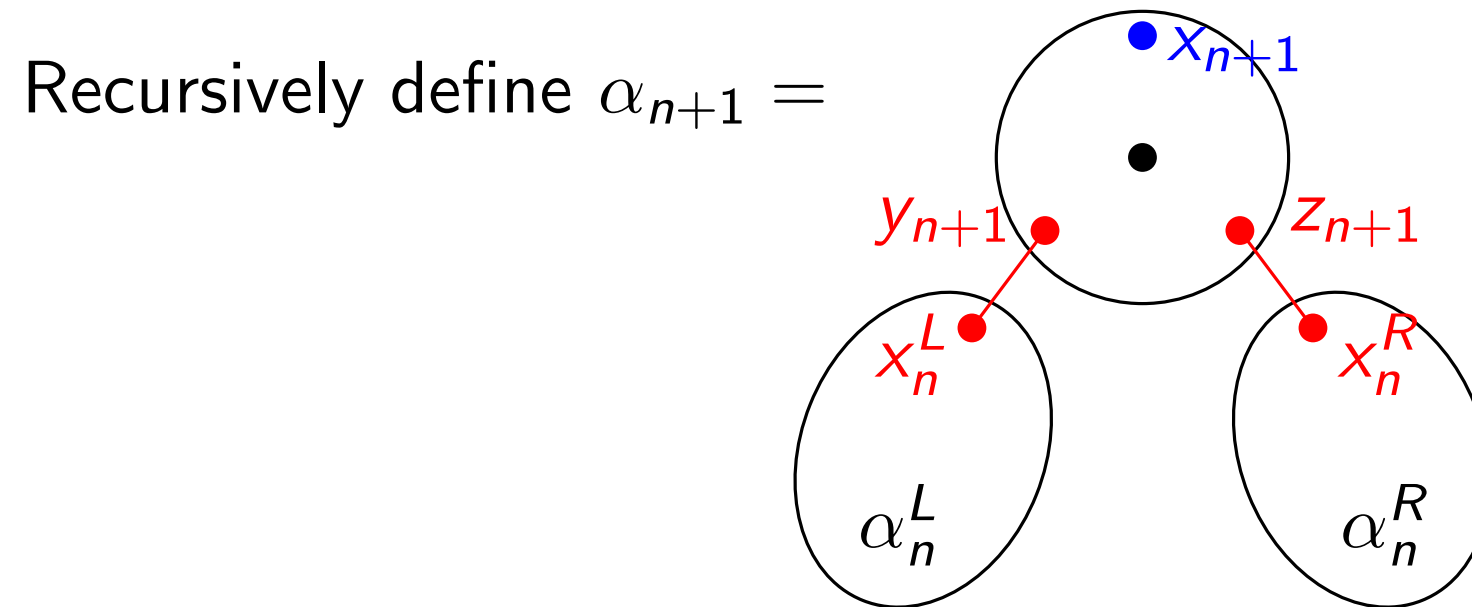
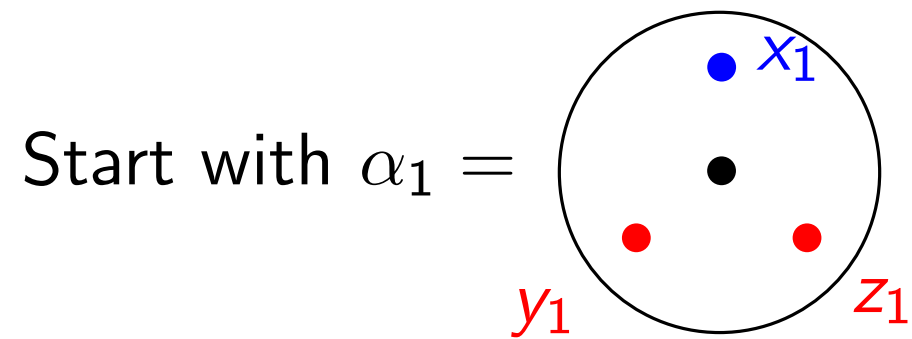
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$$|\mathcal{T}_{\alpha_3}| = Q_3(0) = 1^2 + 12^2 + 67^2 + 224^2 + 484^2 + 672^2 + 571^2 = \mathbf{1066691}$$

Further work: Full binary chain

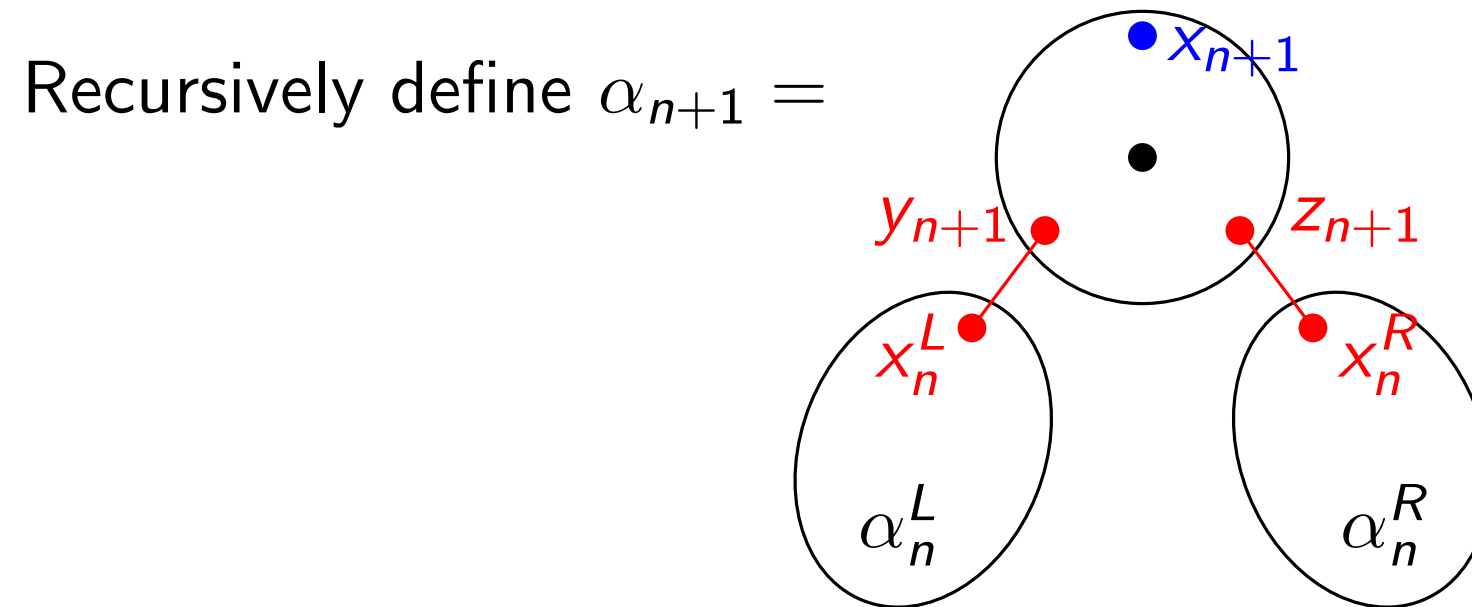
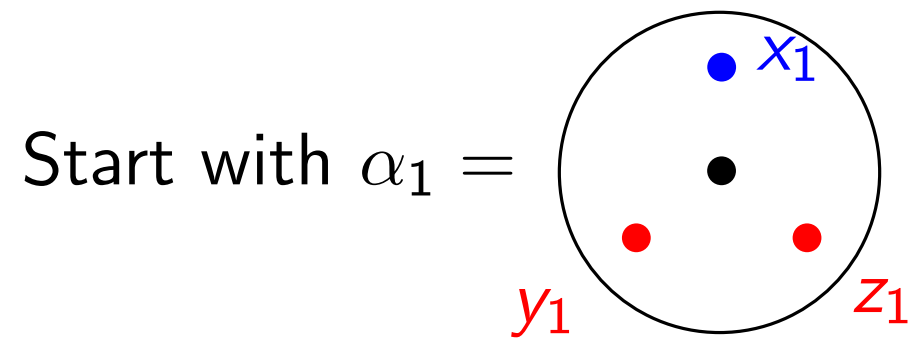


Then if $Q_n(x) = (x + 1)P_n(x + 1)$ and $\Delta(P) = \frac{P - P(0)}{x}$ we have:

$$Q_{n+1}(x) = (x + 1)^4 \sum_{k \geq 2} \left(\Delta^{(k)}(Q_n(x)) \right)^2$$

Makes first terms computation faster **However does not seem to be analytically friendly :(**

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Thank you!