

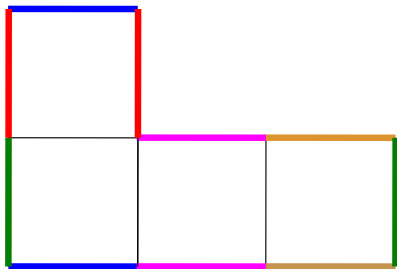
Using Delaunay triangulation to enumerate saddle connections

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supervised by Vincent Delecroix

July 8, 2024

Translation surface

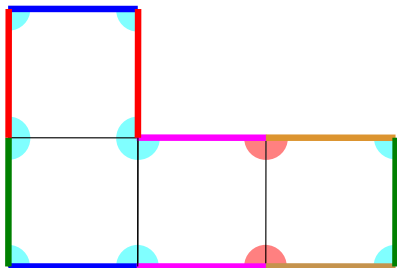
- ▶ Translation surface : polygons glued along their sides by translations.



Genus 2 translation surface

Translation surface

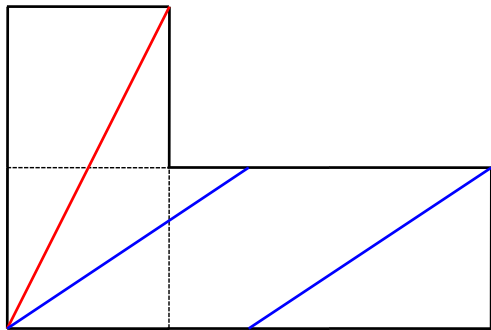
- ▶ Translation surface : polygons glued along their sides by translations.
- ▶ Singularity : point not locally isometric to the euclidian plane.



Genus 2 translation surface with 6π singularity

Saddle connection

A saddle connection is a straight line trajectory whose ends are singularities.



Number of saddle connections

M : translation surface, $R > 0$.

Number of saddle connections of length at most R of M :
 $SC(M, R)$.

Theorem : Masur 1990

$$c_1 R^2 - b \leq |SC(M, R)| \leq c_2 R^2 + b$$

where $c_1 > 0$, $c_2 > 0$ and b are constant depending only of M .

Problem

- ▶ Enumerate saddle connections.
- ▶ Enumerate all saddle connections with length at most R .

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Theorem :

- ▶ Enumeration of saddle connections with $O(1)$ delay.
- ▶ Enumeration of saddle connections with length at most R in $\Theta(R^3)$.
- ▶ In some special cases, enumeration of saddle connections with length at most R in $\Theta(R^2)$.

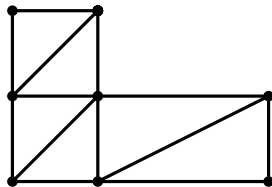
Motivations

- ▶ Geometric : stable norm, length spectra ...
- ▶ Dynamics : flow, ergodicity,...

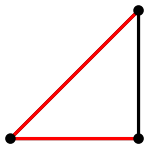
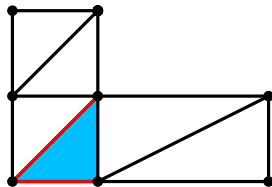
Naïve algorithm

Delaunay tessellation

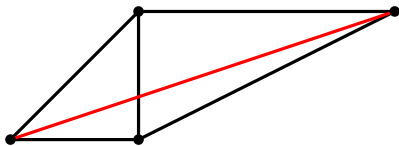
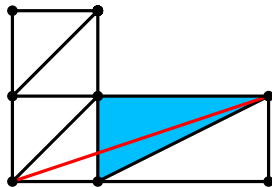
Naive algorithm



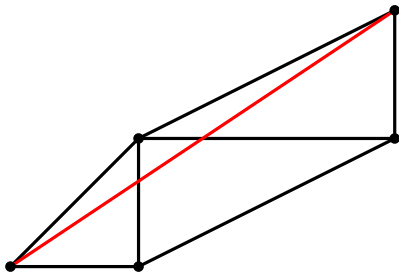
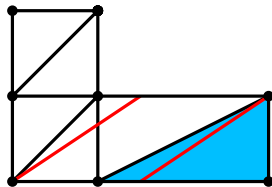
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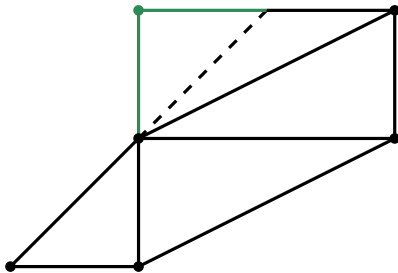
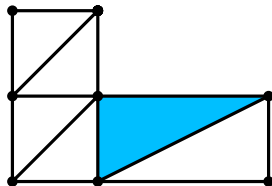
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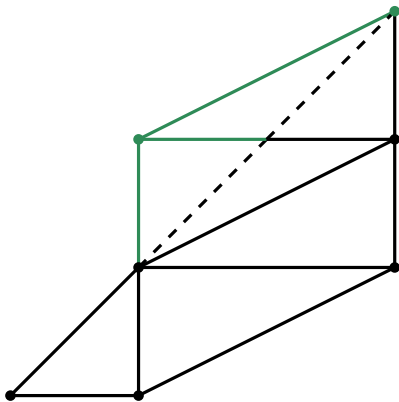
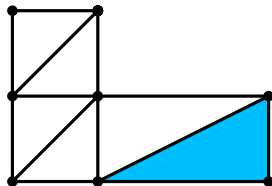
Naive algorithm



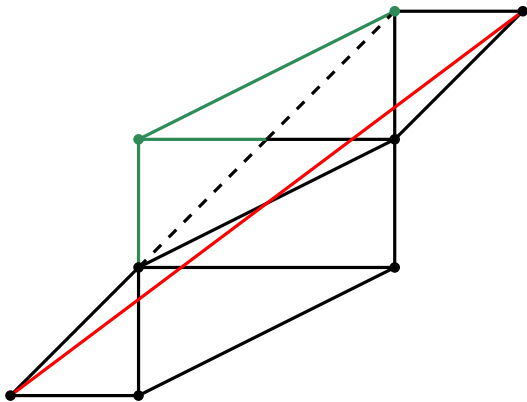
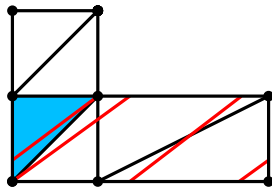
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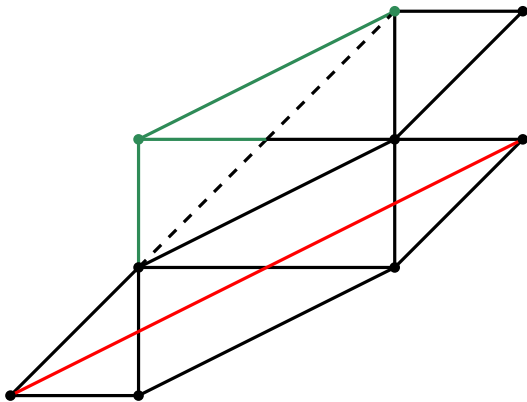
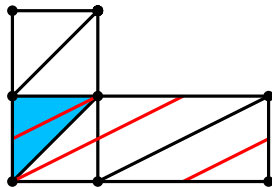
Naive algorithm



Naive algorithm



Naive algorithm



Complexity of the naive algorithm

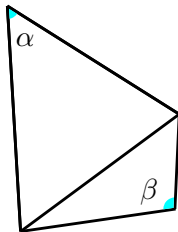
Theorem :

The number of "visible" triangles at distance at most R of z denoted $T(z, R)$ satisfies :

$$k_1 R^3 - bR^2 \leq T(z, R) \leq k_2 R^3 + bR^2$$

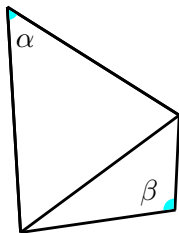
where $k_1 > 0$, $k_2 > 0$ and b are constants depending of M and the choice of triangulation.

Delaunay triangulation

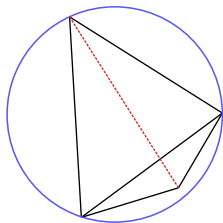
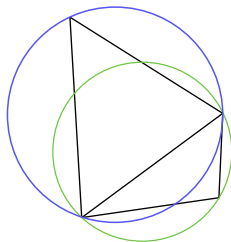


$$\alpha + \beta \leq \pi$$

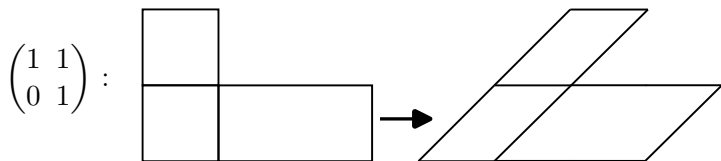
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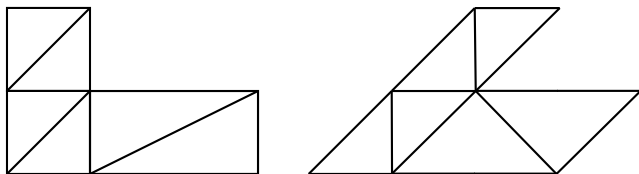
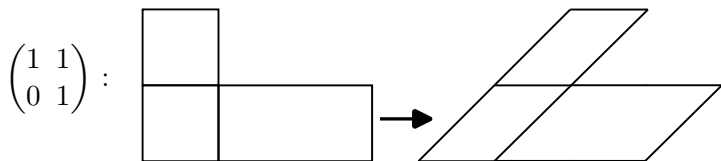
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$SL_2(\mathbb{R})$ -action

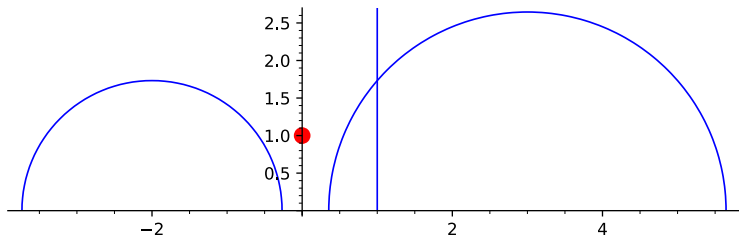


$SL_2(\mathbb{R})$ -action



The hyperbolic plane

- ▶ $\mathbb{H}^2 = \{u + iv \in \mathbb{C} \mid v > 0\} \simeq SO_2(\mathbb{R}) \backslash SL_2(\mathbb{R})$
- ▶ Each point $u + iv \in \mathbb{H}^2$ corresponds to the surface $\frac{1}{\sqrt{v}} \begin{pmatrix} 1 & u \\ 0 & v \end{pmatrix} \cdot M$.



Iso-Delaunay cell

Let τ be a triangulation of M . The iso-Delaunay cell of τ is the set of point of \mathbb{H}^2 admitting τ as a Delaunay triangulation.

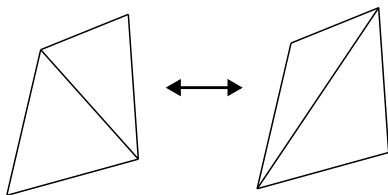
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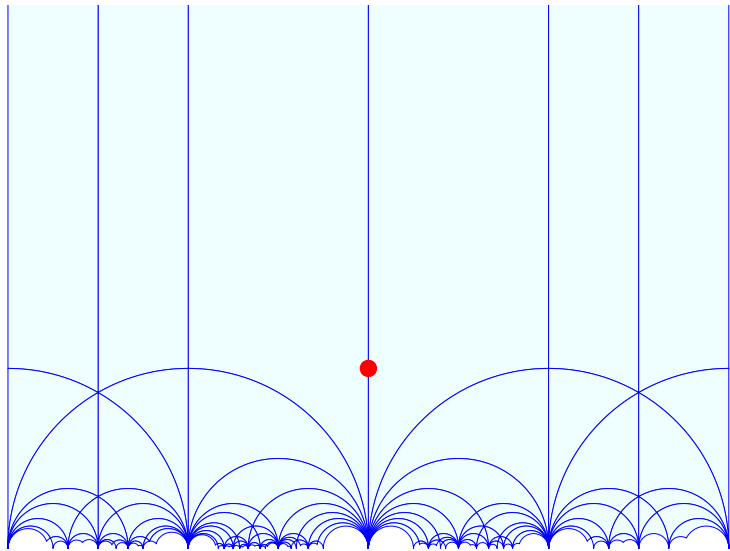
Theorem : Bowman 2008

Each iso-Delaunay cell is a convex polygon of \mathbb{H}^2 with at most $6g - 6 + 3n$ edges.

Moreover each edge correspond to flips.



Iso-Delaunay tessellation



Back to saddle connections

- ▶ Each saddle connection appears in at least one Delaunay triangulation τ (in fact infinitely many).
- ▶ One can enumerate saddle connections with $O(1)$ delay.
- ▶ If M is a Veech surface, one can enumerate saddle connections of length at most R in $O(R^2)$.

Perspectives

- ▶ Enumeration of saddle connections of length at most R in $O(R^2)$.
- ▶ Every complete search over the hyperbolic plane is a $O(1)$ -delay enumeration.