Random Disks of Constant Curvature, Self-Overlapping Curves and Liouville CFT1

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October 21st, 2024

Two dimensional quantum gravity theories come in three different versions:

1) Liouville gravity: consider all metrics on a two dimensional manifold (h,b). Can be formulated rigorously from combinatorics (counting of “maps”, i.e. polygonizations of surfaces) and from the “continuous” point of view (Liouville CFT). This field is so mature that it is now part of probability theory (Sheffield, Duplantier, Vargas, Rhodes, …).

2) Topological gravity: fix both bulk curvature and boundary extrinsic curvature (geodesic boundaries). Amounts to integrating over the moduli space of (h,b) surfaces with the Weil-Petersson measure (Witten, Kontsevitch, Mirzakhani; topological recursion, matrix model).

3) Jackiw-Teitelboim gravity: fix the bulk curvature but do not impose any condition on the boundary. Relevant for near-extremal black holes, holography, related to the duals of SYK and tensor/matrix models, etc. Very interesting on its own as well (see below). Three versions: negative curvature (much studied, holography), zero curvature (simplest set-up for understanding the UV properties) and positive curvature (most interesting? Cosmology).

Action

$$S_{\rm dil} = -\frac{1}{16\pi}\left[\int_{\mathscr{D}} \mathrm{d}^2x\sqrt{g}\,\Phi\Big(R - \frac{2\eta}{L^2}\Big) + 2\Phi_{\rm b}\oint_{\partial\mathscr{D}}\mathrm{d}s\,k\right]$$

$$\int_{\mathscr{D}} \mathrm{d}^2x\sqrt{g}\,R + 2\oint_{\partial\mathscr{D}}\mathrm{d}s\,k = 4\pi = -\frac{2\eta}{L^2}A[g] + 2\oint_{\partial\mathscr{D}}\mathrm{d}s\,k$$

$$S_\Lambda = \frac{\Lambda}{16\pi}A[g] \qquad\qquad \Lambda = -\frac{2\eta\Phi_{\rm b}}{L^2}$$

$$S_{\rm dil} = -\frac{1}{16\pi}\int_{\mathscr{D}} \mathrm{d}^2x\sqrt{g}\,\tilde{\Phi}\Big(R - \frac{2\eta}{L^2}\Big) + \frac{\Lambda}{16\pi}A[g]$$

$$\tilde{\Phi}_{\rm b} = 0$$

$$\eta = \pm 1 \text{ or } 0$$

A strong physical motivation: near-extremal black holes

Basic example: Reissner-Nordstrom in 4D (Kerr could also be studied, etc.)

$$\mathrm{d}s^2 = f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2 \qquad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

$$F_{rt} = \frac{iQ}{r^2}$$

$$r_\pm = M \pm \sqrt{M^2 - Q^2}$$

$$T = \frac{1}{4\pi}\left|f'(r_+)\right| = \frac{1}{4\pi r_+^3}\left(r_+^2 - Q^2\right)$$

$$S = \pi r_+^2$$

Extremal: $M = Q = r_0\,, \quad T = 0$

Near horizon region of the extremal black hole: $r - r_0 = \rho \ll r_0$

$$ds^2 = \frac{\rho^2}{r_0^2}\mathrm{d}t^2 + \frac{r_0^2\mathrm{d}\rho^2}{\rho^2} + r_0^2\mathrm{d}\Omega^2 \qquad F_{rt} = \frac{i}{r_0}$$

$$\mathrm{d}s^2 = r_0^2\Big[\frac{\mathrm{d}t^2 + \mathrm{d}z^2}{z^2} + \mathrm{d}\Omega^2\Big] \qquad z = \frac{r_0^2}{\rho}$$

The near-horizon region is $\mathrm{H}_2 \times \mathrm{S}^2$

The geometry develops a long hyperbolic (AdS) throat.

The fundamental puzzles

In the near extremal limit, i.e. T goes to zero, one finds that:

$$M = E = r_0 + 2\pi^2 r_0^3 T^2 + \cdots$$

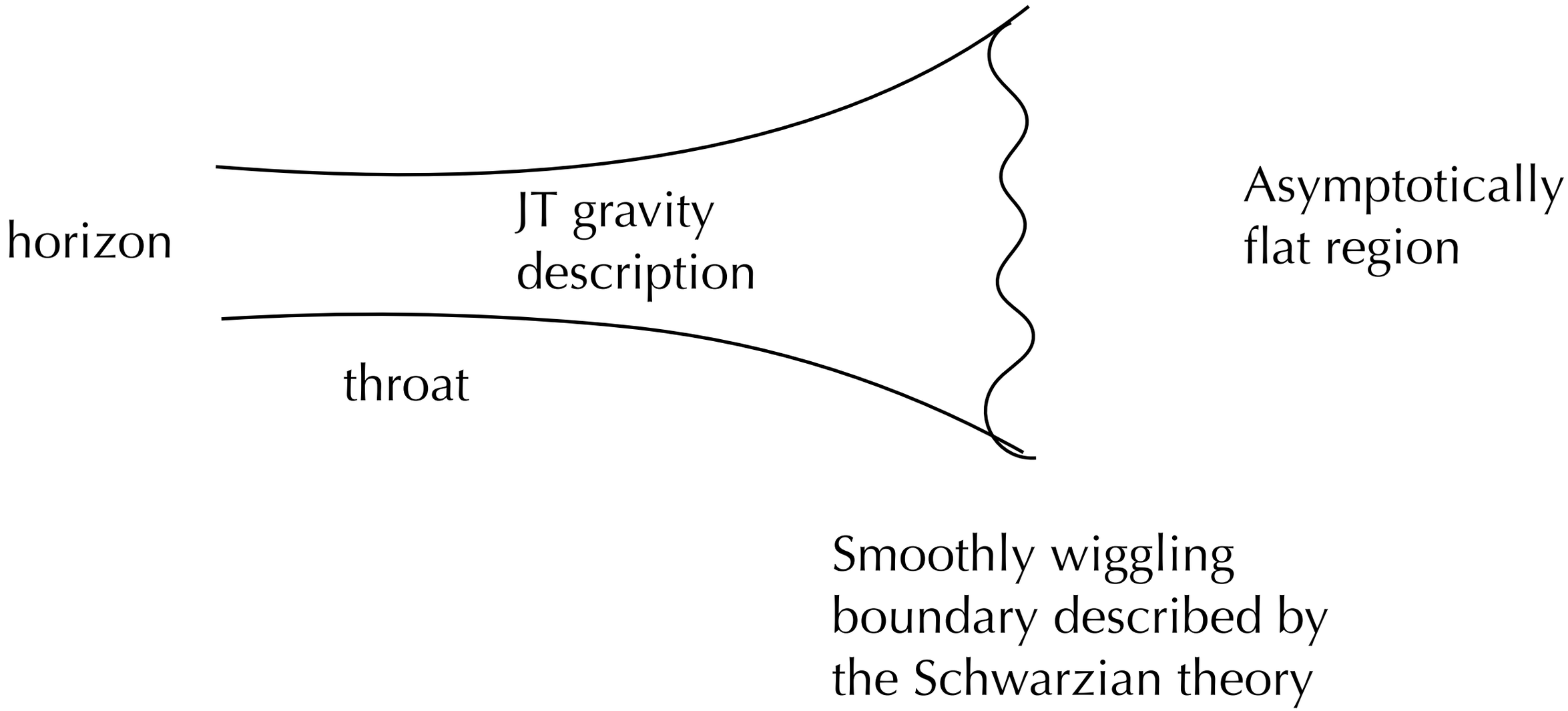
$$S = \pi r_0^2 + 4\pi^2 r_0^3 T + \cdots$$

This semi-classical analysis predicts:

1) A huge ground state degeneracy (extremely suspicious for a non-supersymmetric system)
2) An inconsistency with the usual Hawking evaporation process at as soon as $T \lesssim 1/r_0^3$

The resolution of the puzzle shows that the semi-classical analysis breaks down at very low temperature in spite of the fact that the geometry is arbitrarily weakly curved, due to strong quantum fluctuations of a metric mode in the near-horizon region.

The near-horizon region turns out to be described by JT gravity in negative curvature in a certain limit (Schwarzian).



The boundary between the near-horizon region and the rest of the geometry fluctuates and the fluctuations correspond to particular geometries in JT that we describe in more detail later.

One can take into account these fluctuations exactly. They yield the crucially needed modifications in the low-temperature energy and entropy:

Naive semi-classical:

$$M = E = r_0 + 2\pi^2 r_0^3 T^2 + \cdots \qquad S = \pi r_0^2 + 4\pi^2 r_0^3 T + \cdots$$

Taking into account the Schwarzian mode:

$$M = E = r_0 + \frac{3}{2} T + 2\pi^2 r_0^3 T^2 + \cdots \qquad S = \pi r_0^2 + 4\pi^2 r_0^3 T + \frac{3}{2} \ln T + \cdots$$

The correction to the entropy is associated with a density of state

$$\rho(E) \propto \sinh\left(c\sqrt{E - r_0}\right)$$

So, on top of being an extremely interesting model on its own (I hope you'll be convinced by the end of the talk!), recall that JT in negative curvature and in some limit describes a fundamental quantum gravity effect in ``real-world'' black holes.

JT in zero or positive curvature is also very relevant for physics, describing the near-extremal black holes in de Sitter space and, even more importantly, toy cosmological models.

But essentially nothing was known about these models before my recent work, because they do not have a Schwarzian description. We shall talk more about these models later on.

Goals of this research

Formulate Euclidean Jackiw-Teitelboim quantum gravity in a rigorous way, for finite value of the parameters (and thus away from the Schwarzian limit of the negative curvature model; and for the zero and positive curvature models as well).

We would like to eventually put the subject on the same footing as Liouville gravity.

Two points of view : combinatorial and probabilistic (“discrete” and “continuous”)

We are also interested in the real-time version of the theory, which comes with its load of additional puzzles (not for today!)

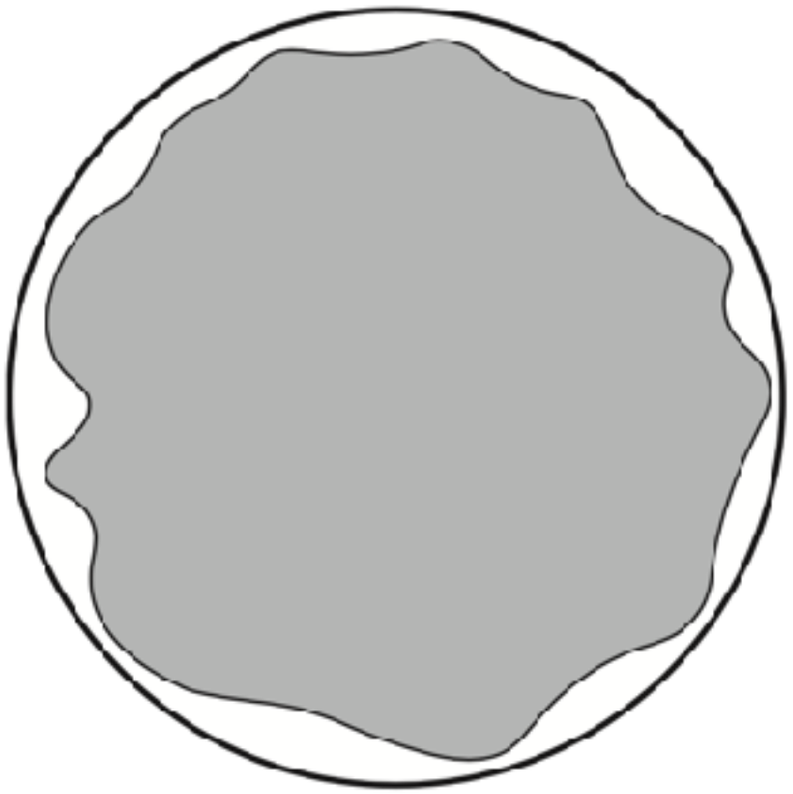
Upshots

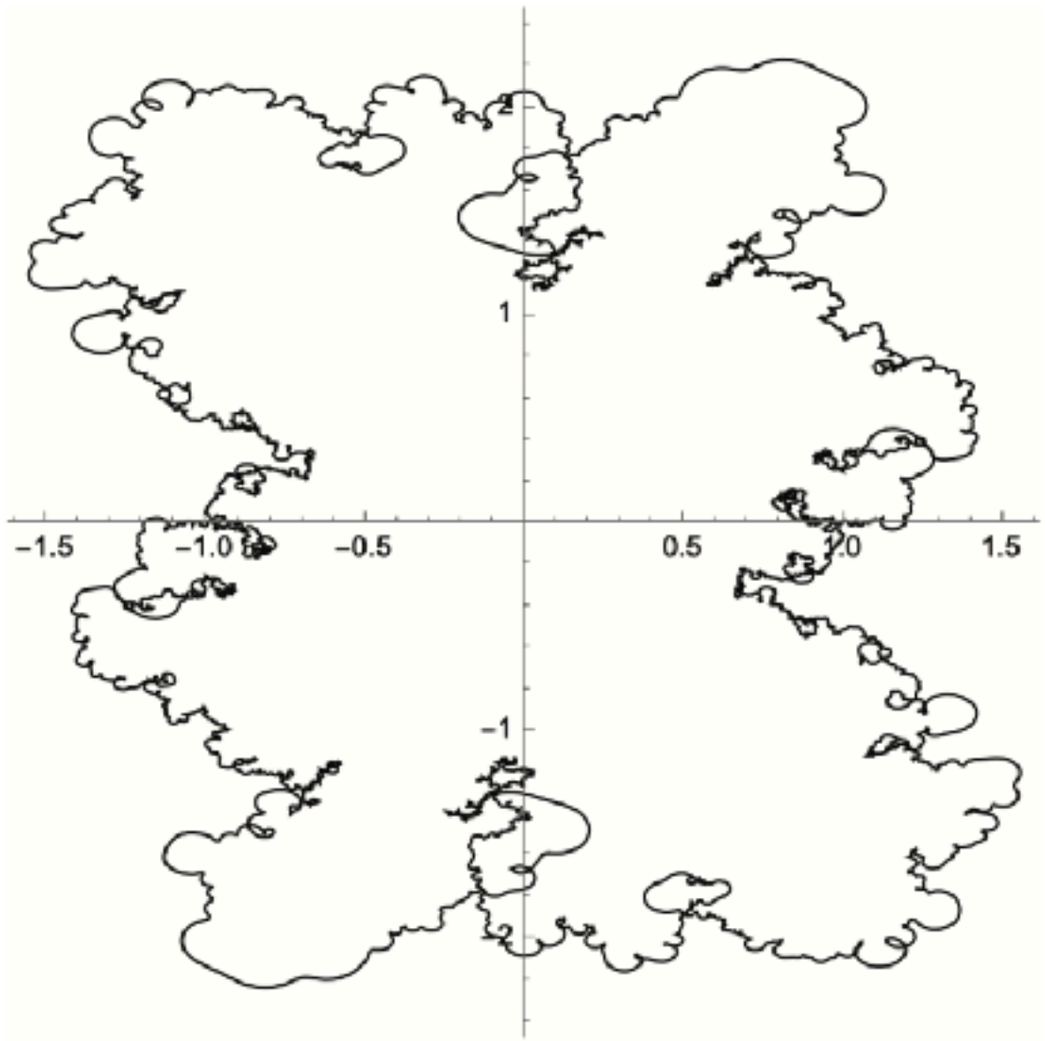
1) The usual formulation of the negative curvature model in terms of the Schwarzian theory is approximate. The Schwarzian description is an effective, long distance description of JT gravity, valid on distance scales that are much larger than the curvature length scale. In particular, the Schwarzian description plays a role in JT gravity which is similar to the role it plays in the SYK model.

2) A discretized formulation of JT gravity is possible in terms of a new model of random path (describing the fluctuating boundary). The model has remarkable, surprising properties. A matrix model doing the correct counting of configurations can be constructed (not solved, yet).

3) All three models (negative, zero and positive) curvatures have the same short-distance properties. But their long-distance properties are qualitatively different. The Schwarzian is only relevant to describe the long-distance behaviour of the negative curvature model. The positive curvature model exists only for positive cosmological constant.

4) The technology developed in the context of Liouville gravity can be imported in JT gravity as well. The conformal gauge can be used to construct an explicit, rigorous and a priori tractable framework within which the theory can be studied.





Reparameterization ansatz/
smooth and gently wiggling boundary/
Schwarzian description

Typical configuration/
fractal boundary/
description in terms of a distribution-valued boundary conformal factor

References

F. F., Jackiw-Teitelboim Gravity, Random Disks of Constant Curvature, Self-Overlapping Curves and Liouville CFT_1, arXiv:2402.08052 (synthetic letter)

F. F., Random Disks of Constant Curvature: the Lattice Story, arXiv:2406.06875

F. F., Random Disks of Constant Curvature: the Conformal Gauge Story (to appear)

The semi-classical limit, using the framework developed above, has been studied in:

S. Chaudhuri and F. F., Finite cut-off JT and Liouville quantum gravities on the disk at one loop, arXiv:2404.03748

S. Chaudhuri and F. F., Dirichlet Scalar Determinants On Two-Dimensional Constant Curvature Disks, arXiv:2405.14958

More work in progress with Vincent Vargas, Baptiste Cerclé, Soumyadeep Chaudhuri…

Previous works on attempts to a microscopic definition of JT

Brownian bridge:

A.Kitaev and S.J. Suh, ``Statistical mechanics of a two-dimensional black hole,'' JHEP 05 (2019) 198, arXiv:1808.07032 [hep-th].

Self-avoiding loops:

D. Stanford and Z. Yang, ``Finite-cutoff JT gravity and self-avoiding loops,'' arXiv:2004.08005 [hep-th].

TTbar:

L.V. Iliesiu, J. Kruthoff, G.J. Turiaci and H. Verlinde, ``JT gravity at finite cutoff,'' SciPost Phys. 9 (2020) 023, arXiv:2004.07242 [hep-th].

Plan

1) Discretized formulation and self-overlapping polygons

2) Discussion of the macroscopic properties (emergence of the Schwarzian description in negative curvature, breakdown of the positive curvature theory for negative cosmological constant, …)

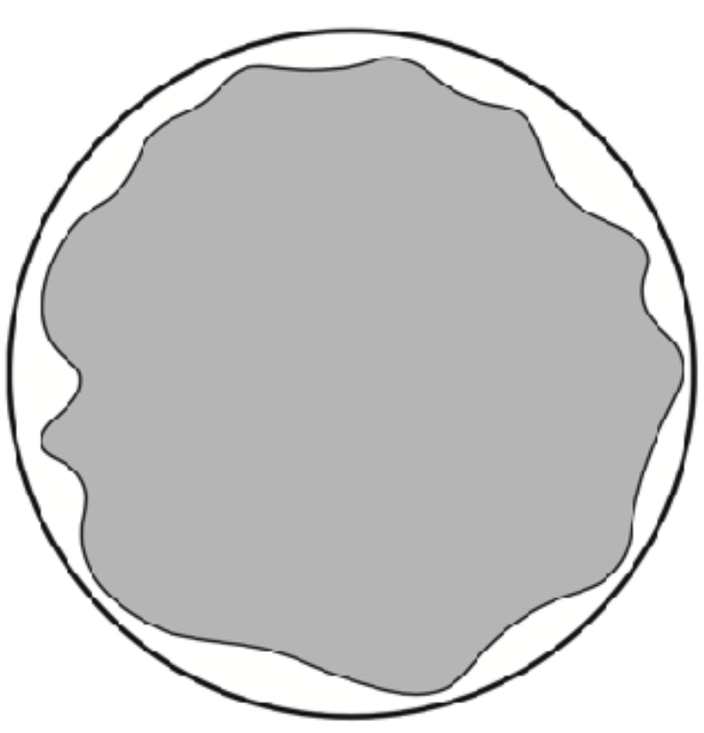
3) Conformal gauge: conceptual discussion; semi-classical approximation; exact description and first exact results.

$$\int_{\mathrm{Met}^{0,\pm}_{\beta_{\mathrm{q}}}(\mathscr{D})} \mathrm{d}\mu(g)\, e^{-\frac{\Lambda}{16\pi}A[g]}$$

The metric space for finite geometries

It can be described using immersions of the disk into a canonical space (hyperbolic space, Euclidean space or the two-sphere). This yields a ``distorted disk'' representing isometrically a metric on the original ``source'' disk.

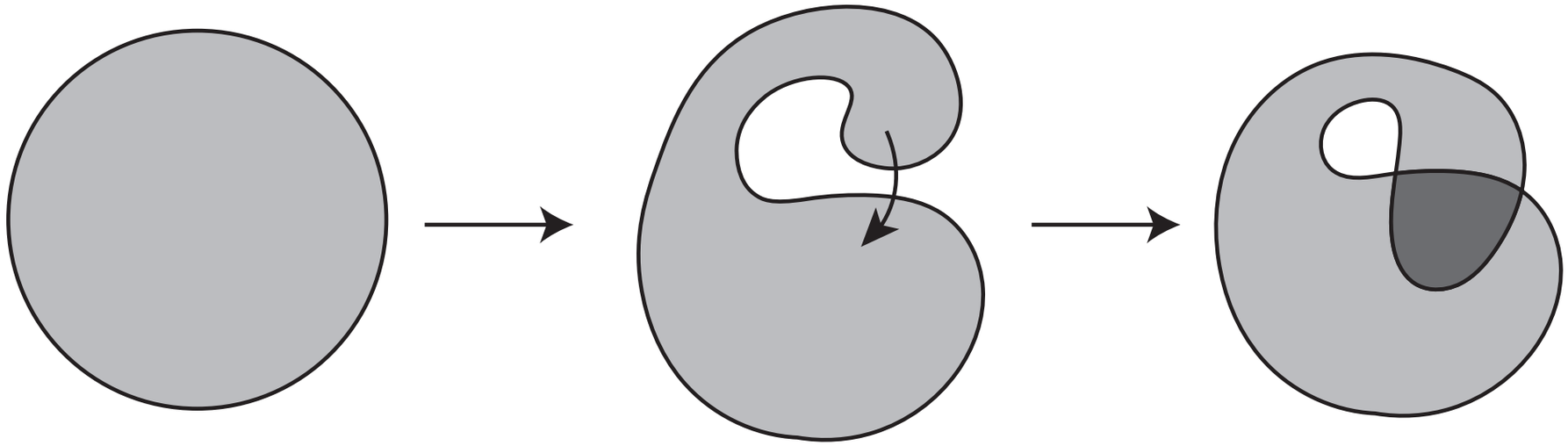
$$F : \mathscr{D} \to \mathbb{R}^2 \qquad\qquad g = F^* \delta$$



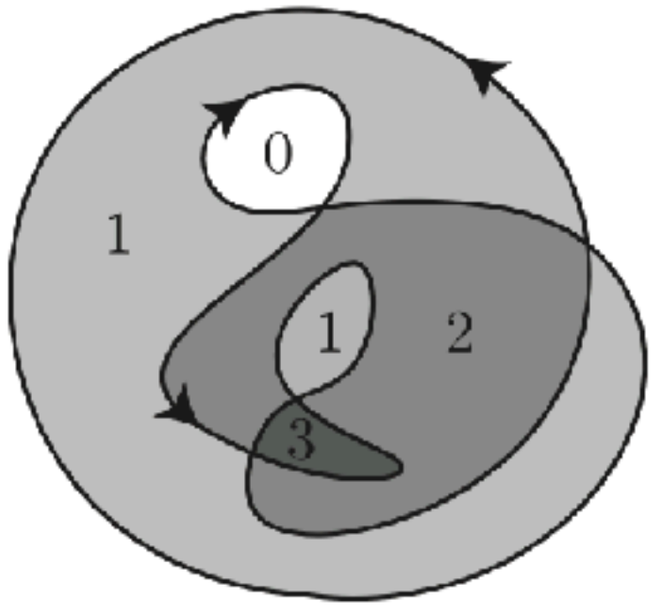
reparameterization ansatz

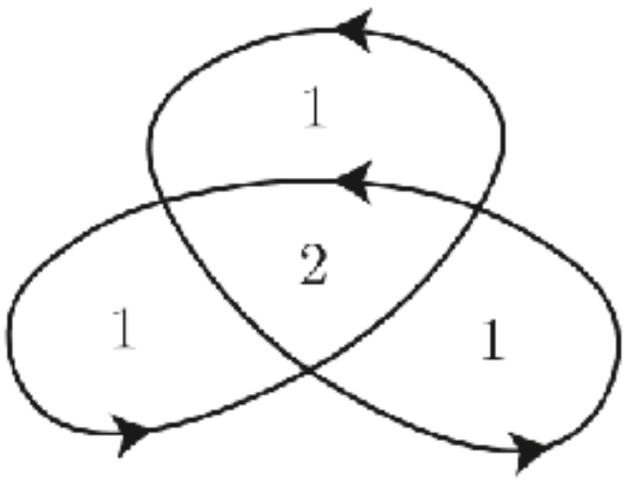
Self-avoiding loop

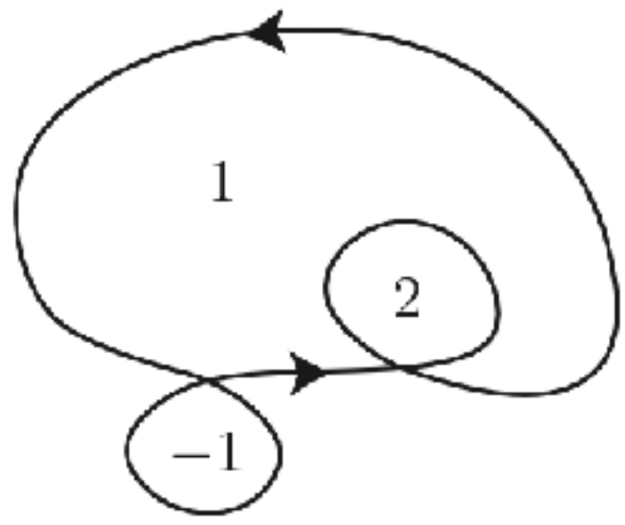
Overlaps



Winding numbers

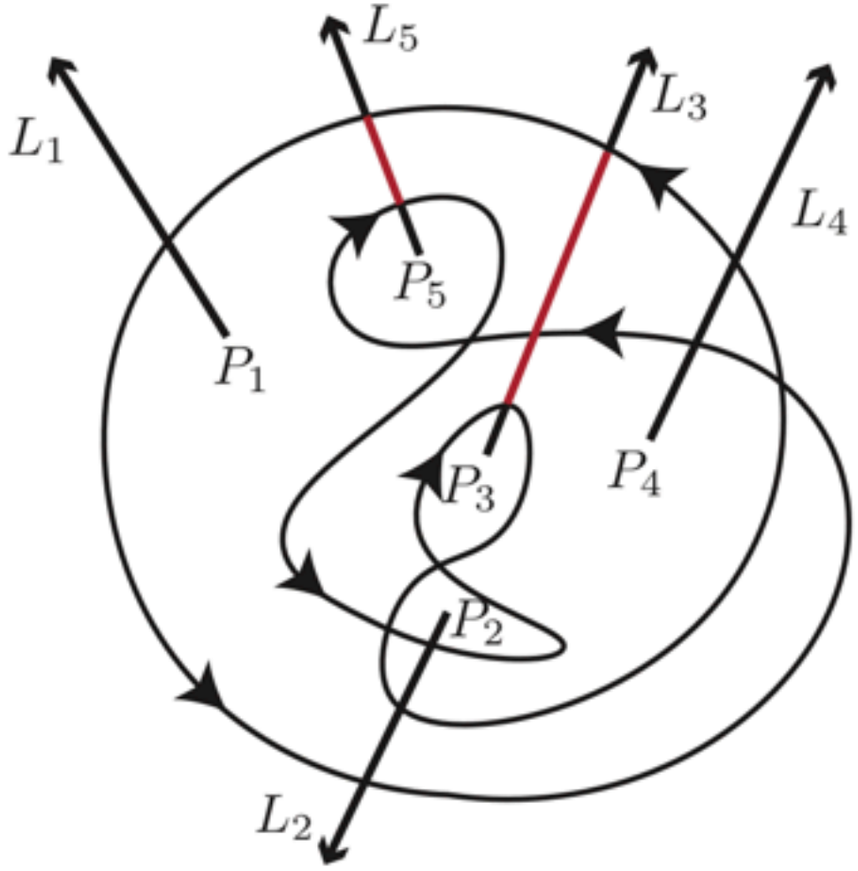


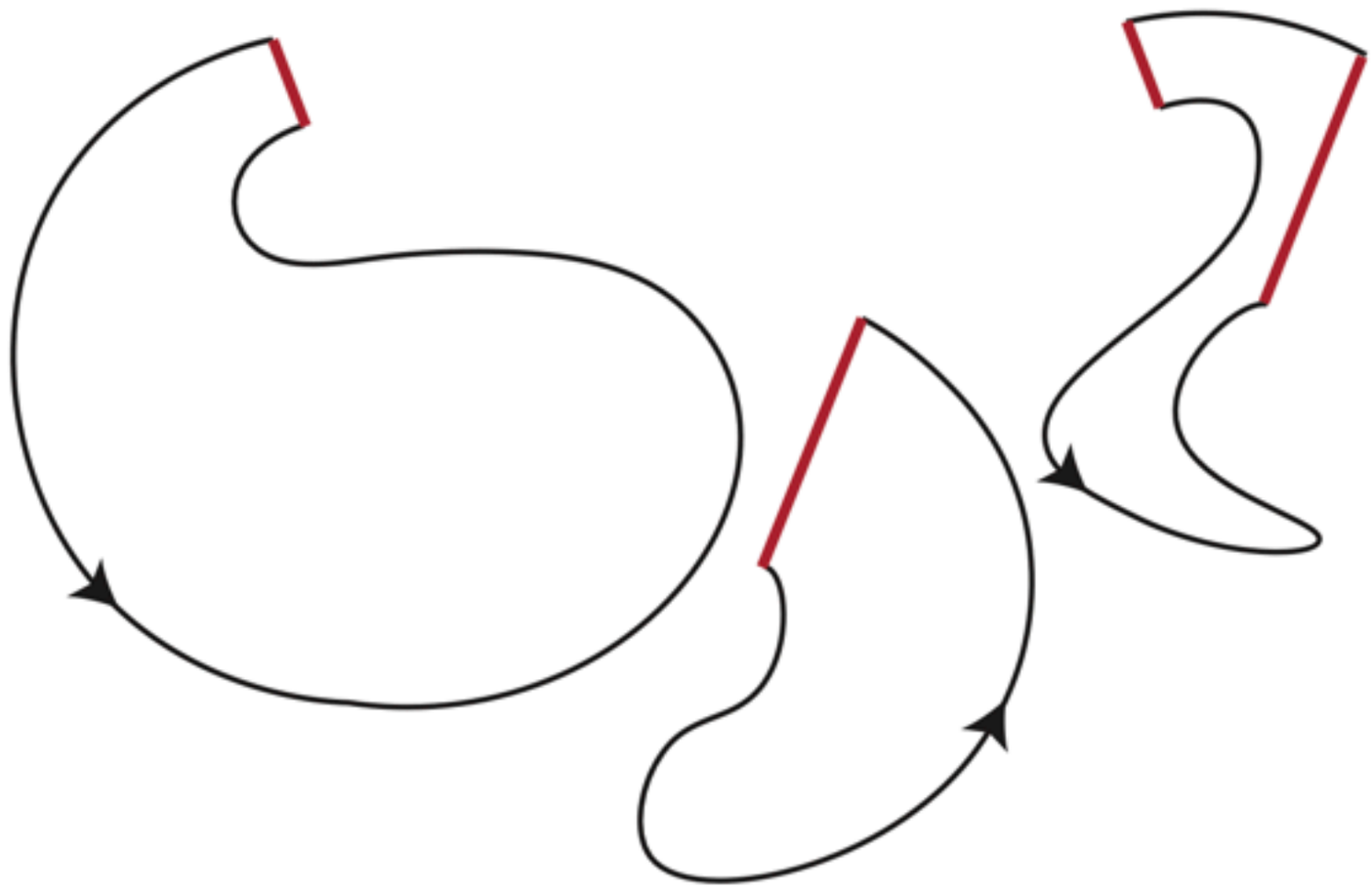


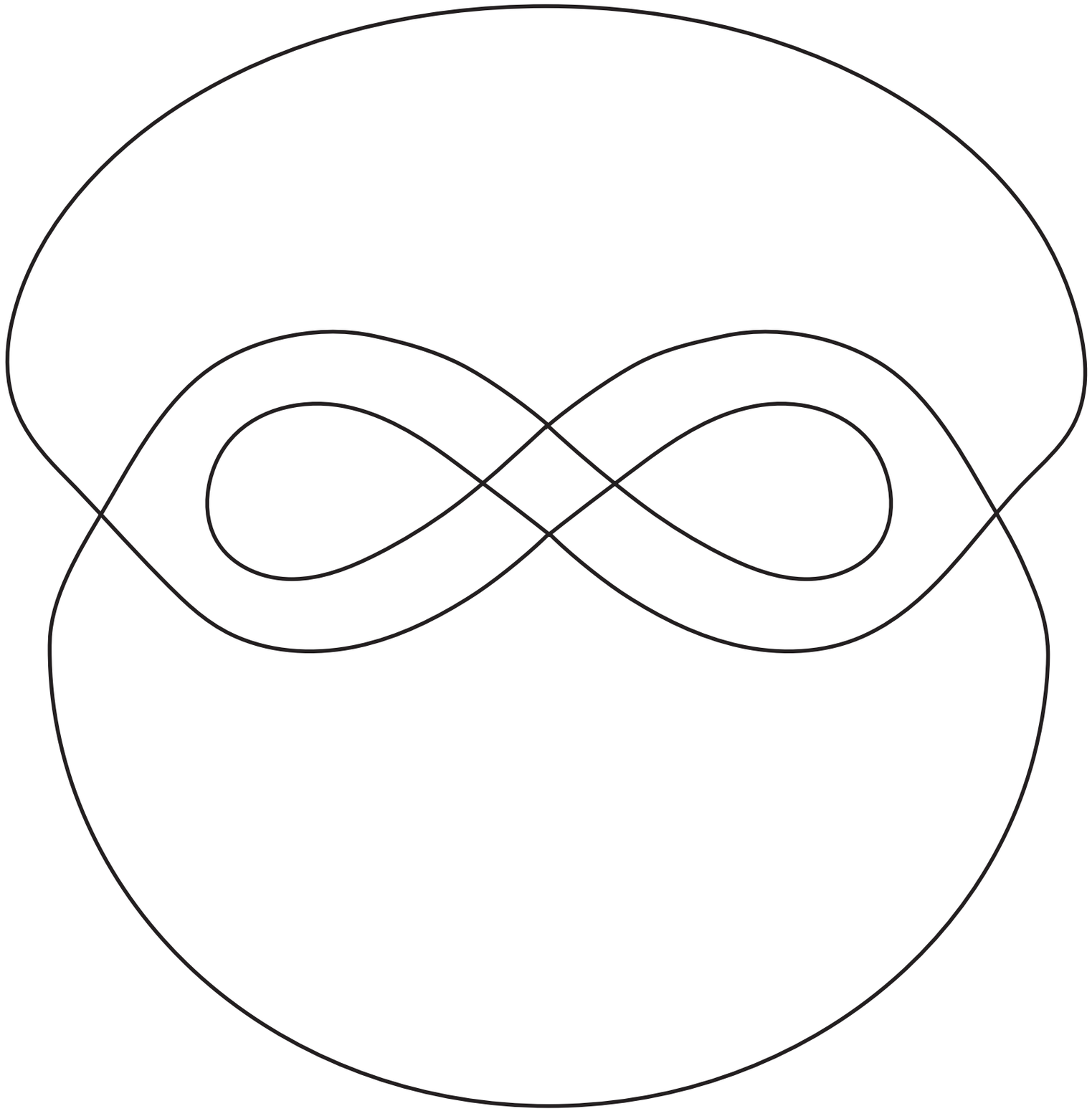


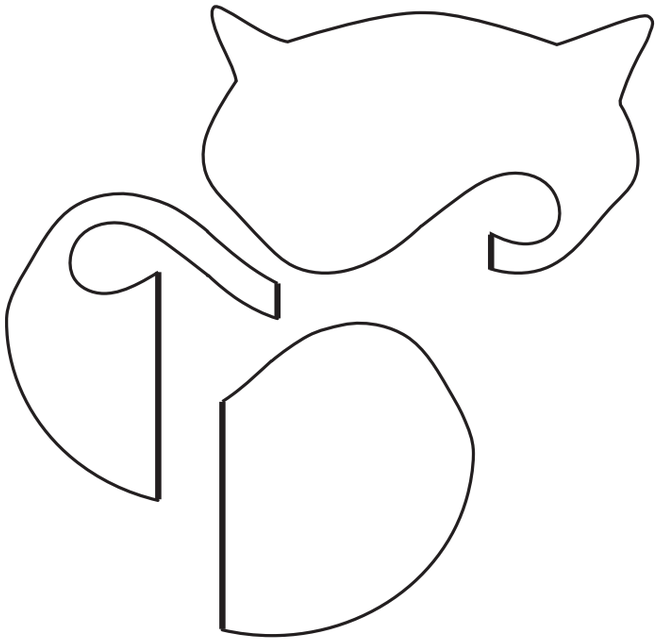
Blank cuts

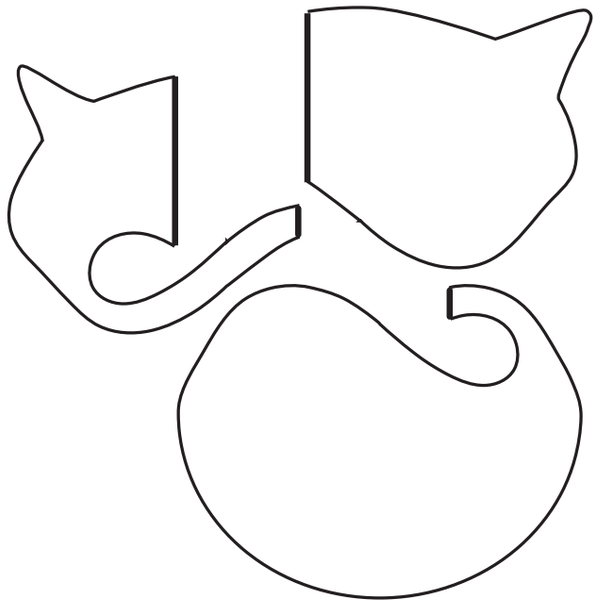
C. Titus ``The combinatorial topology of analytic functions of the boundary of a disk,'' Acta Mathematica 106(1) (1961) 45
S.J. Blank, ``Extending Immersions of the Circle,'' Ph.D.\ dissertation, Brandeis University (1967).
V. Poénaru, ``Extension des immersions en codimension 1,'' Astérisque 10 (1966--1968) 473 (Bourbaki seminar number 342).
P.W. Shor and C.J. Van Wyk, ``Detecting and decomposing self-overlapping curves,'' Computational Geometry 2 (1992) 31.

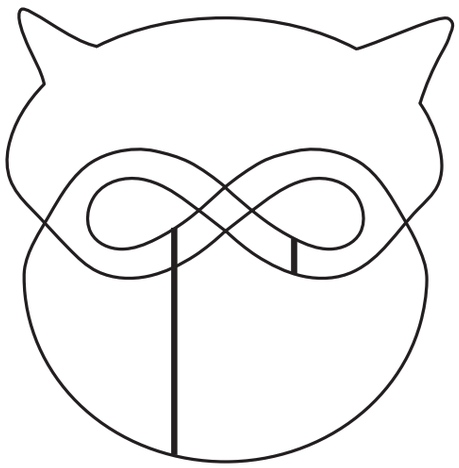


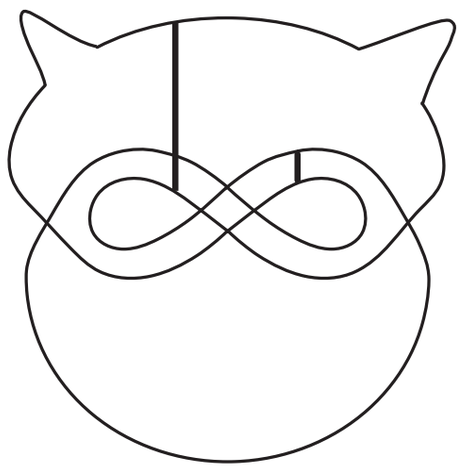


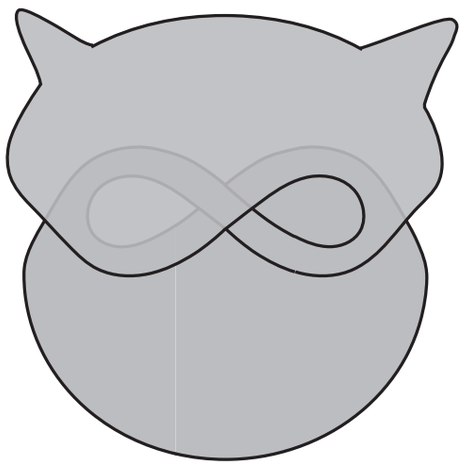


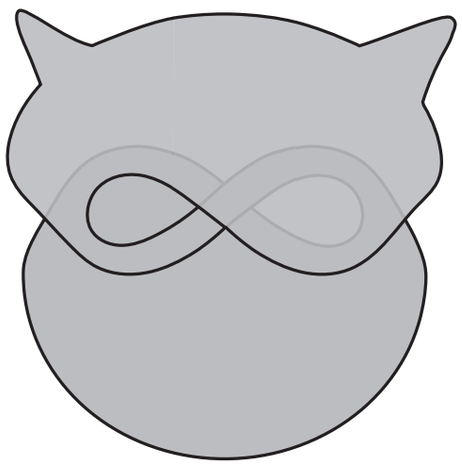






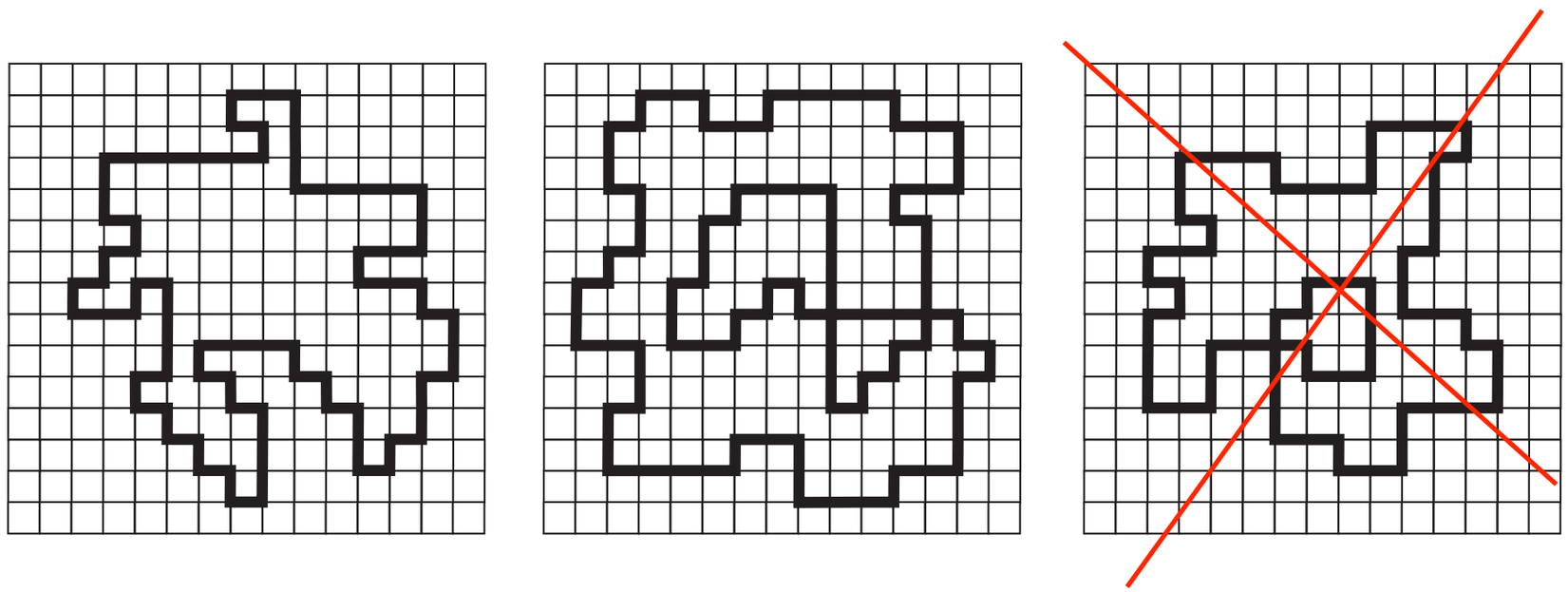






A given boundary curve may bound several distincts disks (i.e. be associated with several distinct metrics on the source disk!) -> multiplicity index

Upshot: JT gravity is a random curve model of a new type. We have to count self-overlapping curves, taking into account the multiplicity.



Precise counting

1) path code: Straight, Right of Left move, modulo cyclic permutation and reversal.

2) We do not identify configurations related by reflections (though we could, but it would be unnatural from the matrix model perspective).

3) A typical configuration is counted 4 times, corresponding to acting with lattice rotations. Symmetric configurations are counted 2 times or 1 time.

4) We also need (crucially) to take into account the “Milnor” multiplicity.

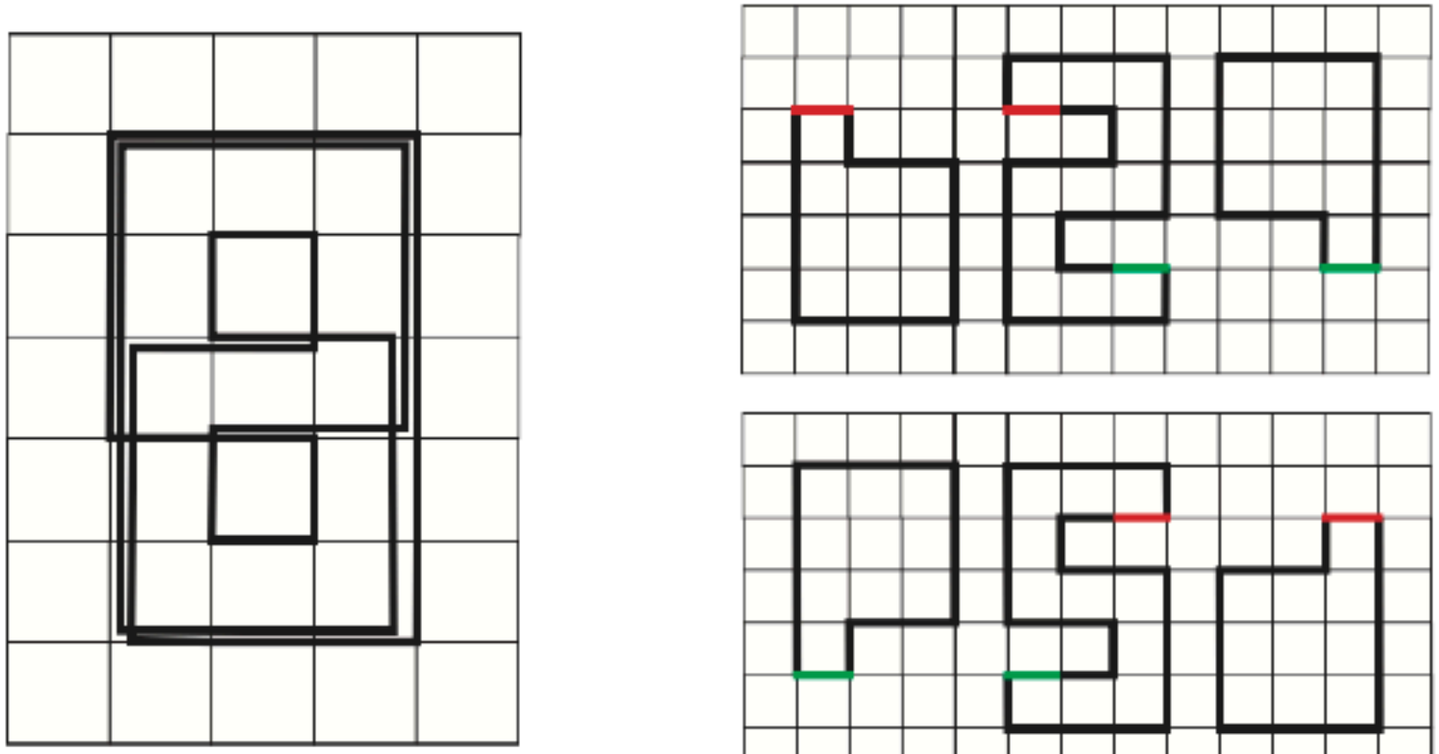


Figure 14: The smallest SOP with multiplicity two on a square lattice, with boundary code SSSSLSSLSSLSRRRRSLSSLSSLSSSSLSSLSSLSRRRRSLSSLSSL, $2n = 48$ and $p = 31$. This configuration is $\mathbb{Z}_2$-symmetric (invariant under a 180 degree rotation) and has multiplicity 2. It is thus counted $2 \times 2 = 4$ times in the generating function. Similar Milnor configurations with no symmetry can be easily obtained, for instance by adding a face at the bottom, with new boundary code SSSSLSSLSSLSRRRRSLSSLSSLSSSSLSSLSSLSRRRRSLSSL**RLLR**L (the added sequence is in boldface). Such a configuration must be counted 8 times in the generating function.

Generating function and scaling

$$W(t,g) = \sum_{\gamma} \mu_\gamma t^{F(\gamma)} g^{v(\gamma)} = \sum_{n,p\geq 1} W_{2n,p} t^p g^{2n}$$

Typical expectations:

$$w_{2n;r} = \sum_{p\geq 1} p^r W_{2n,p} \underset{n\to\infty}{\sim} a_r g_*^{-2n} (2n)^{2\nu(r-\vartheta)-1}$$

$$\langle p^r \rangle_{2n}(t=1) = \frac{w_{2n;r}}{w_{2n}} \underset{n\to\infty}{\sim} \frac{a_r}{a_0} (2n)^{2\nu r}$$

Precise conjecture for the SOP model:

$$\nu = \frac{1}{2}, \quad \vartheta = 2$$

$$w_{2n} \underset{n\to\infty}{\sim} a_0 g_*^{-2n} (2n)^{-3} (\ln n)^{\varphi}$$

Continuum limit $n \to \infty\,, \quad \ell_0 \to 0\,, \quad 2n\ell_0^{1/\nu} = \beta_{\rm q}$ fixed.

$$\langle A^r \rangle \sim n^{2\nu r}$$

$\frac{1}{\nu} = d_{\rm H}$ is the Hausdorff dimension.

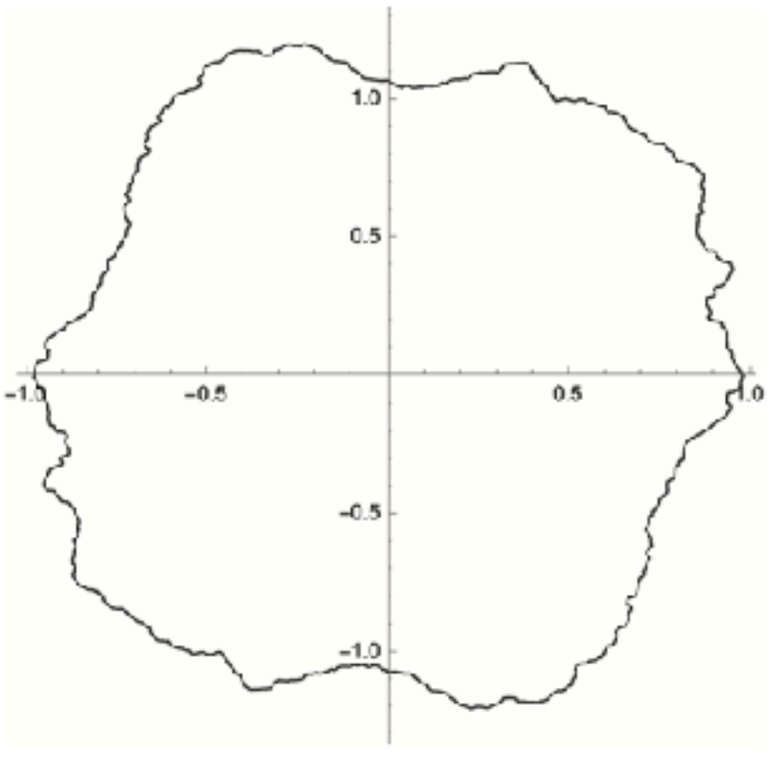
$\beta_{\rm q}$ is the quantum boundary length.

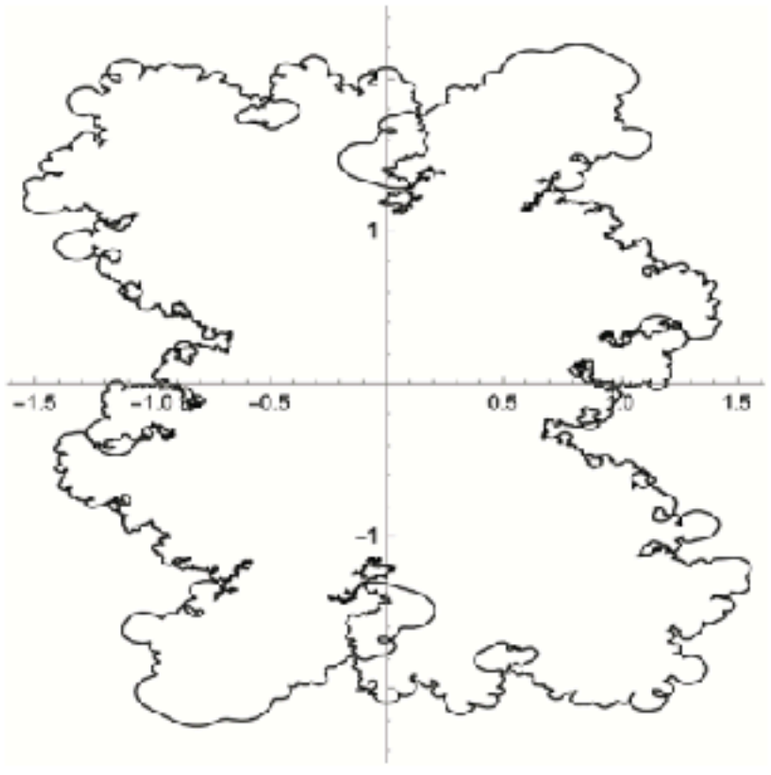
$$\langle A^r \rangle = \int_0^\infty A^r \rho_{\beta_{\rm q}}(A) {\rm d}A$$

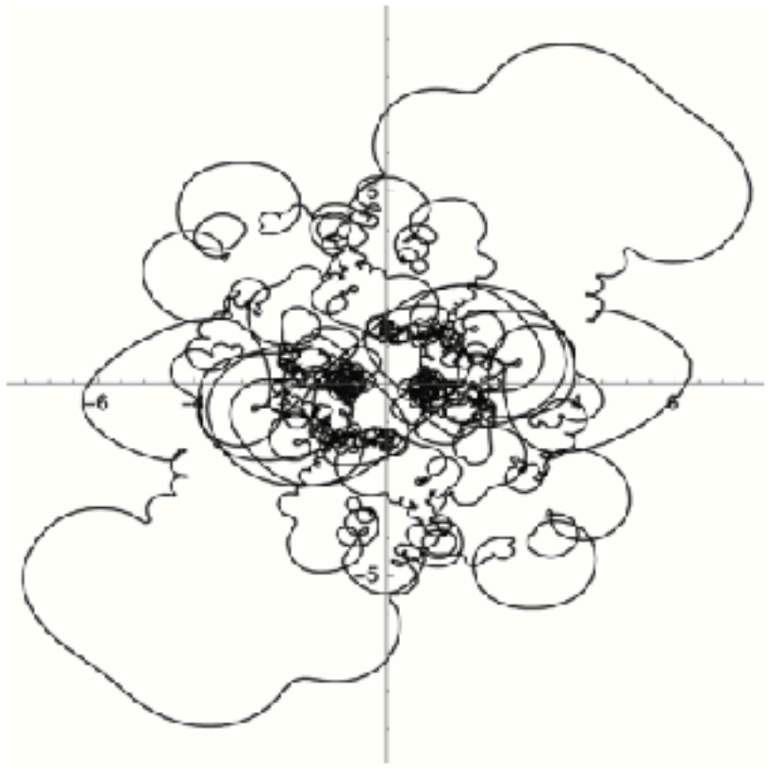
In JT gravity coupled to conformal matter of central charge c,

$$\nu = \frac{1}{2}\left[1 + \sqrt{\frac{c}{c-24}}\right]$$

There is a c=0 barrier (analogous to the c=1 barrier in Liouville)







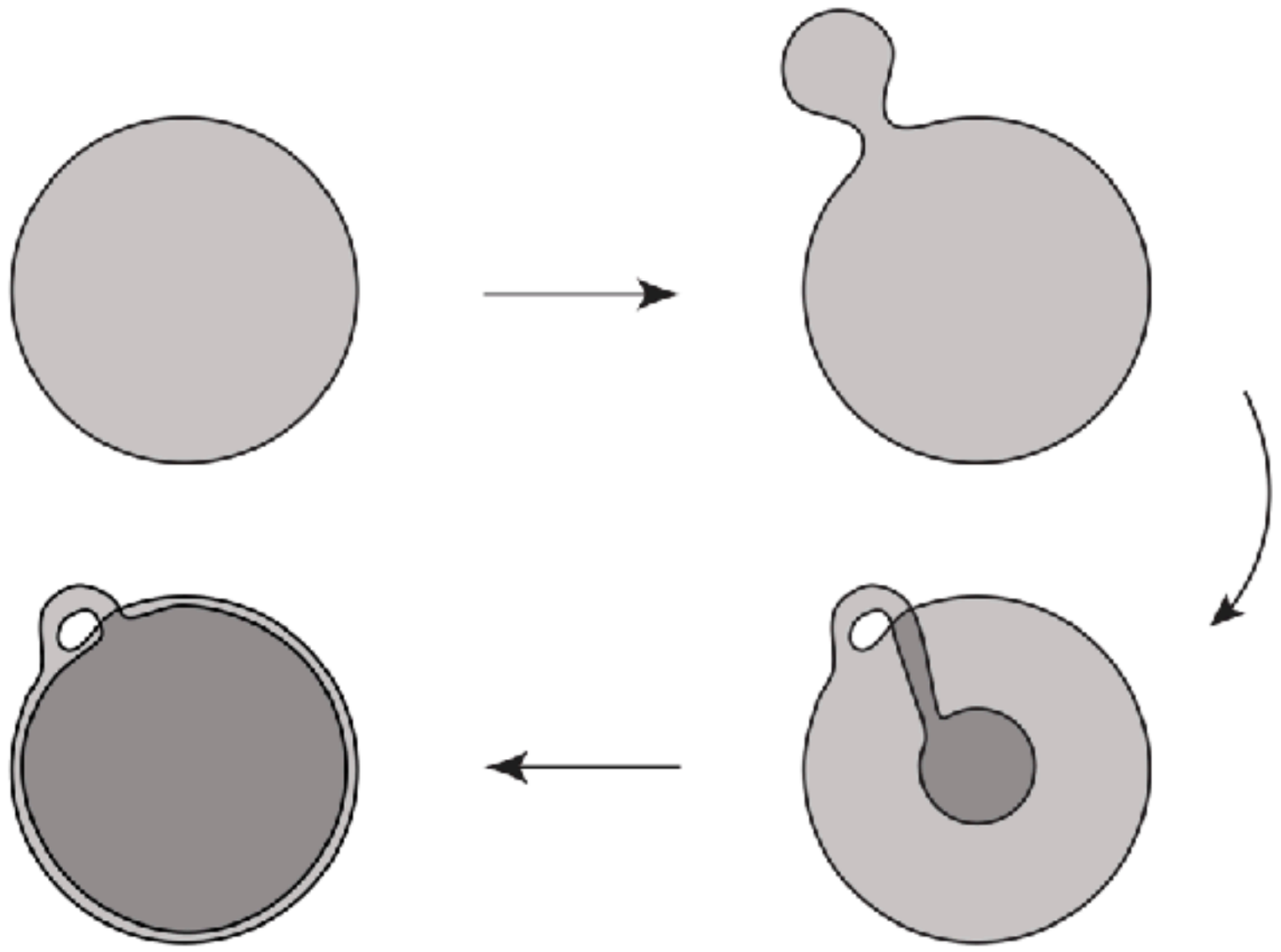
$c = -125$ $c = 0$ $c = 12$

Long distance properties, quantum version of the isoperimetric inequalities

$$A \leq 2\pi L^2 \left[-1 + \sqrt{1 + \frac{\ell^2}{4\pi^2 L^2}} \right] \sim \ell L \quad \text{when } R = -2/L^2.$$

$$A \leq \frac{\ell^2}{4\pi} \quad \text{when } R = 0$$

No such inequality in the case of positive curvature (see below).



One step in the construction of a metric of constant positive curvature on the disk having arbitrarily large area and arbitrarily small boundary length.

So one expects large area to be highly improbable in negative curvature, improbable in zero curvature and ubiquitous in positive curvature.

This yields quantum versions of the isoperimetric inequalities.

Conjecture 4.5. *The area distribution of the SOP model, i.e. pure JT gravity in zero curvature, decays exponentially at large area, with*

$$\ln \rho_{\beta_q}^{SOP}(A) \underset{A\to+\infty}{\sim} -\kappa A/\beta_q\,, \tag{4.118}$$

where κ is a strictly positive numerical constant.

Conjecture 4.6. *The area distribution of JT gravity in zero curvature coupled to $c \leq 0$ conformal matter, which has a critical exponent ν given by Eq. (4.57), has the asymptotic behaviour*

$$\ln \rho_{\beta_q}^{0}(A) \underset{A\to+\infty}{\sim} -\kappa_\nu \left(A/\beta_q^{2\nu}\right)^{\chi_\nu}, \tag{4.119}$$

where κ_ν is a strictly positive numerical constant and χ_ν a critical exponent, which is a strictly increasing function of ν satisfying $\lim_{\nu\to\frac{1}{2}+}\chi_\nu = 1$ and $\lim_{\nu\to 1-}\chi_\nu = +\infty$.

$$\chi_\nu = \frac{1}{2(1-\nu)}\,.$$

As a consequence, the zero curvature theory exists only for a range of cosmological constants, $\Lambda > \Lambda_c$ for $\Lambda_c < 0$.

$$W(\Lambda) = W(0)\int_0^\infty \rho(A) e^{-\frac{\Lambda}{16\pi}A}\,\mathrm{d}A$$

Conjecture 6.1. *The area distribution of JT gravity in negative curvature coupled to $c \leq 0$ conformal matter, with critical exponent ν given by Eq. (4.57), has the asymptotic behaviour*

$$\ln \rho^{-}_{\beta_q}(A) \underset{A\to+\infty}{\sim} -\kappa^{-}_{\nu}\left(\frac{A}{\beta_q^{\nu}L}\right)^{\chi^{-}_{\nu}}, \tag{6.4}$$

where κ^{-}_{ν} is a strictly positive numerical constant and χ^{-}_{ν} a critical exponent, which is a strictly increasing function of ν satisfying $\lim_{\nu\to\frac{1}{2}+}\chi^{-}_{\nu}=2$ and $\lim_{\nu\to 1-}\chi^{-}_{\nu}=+\infty$.

$$\chi^{-}_{\nu}=\frac{1}{1-\nu}\,.$$

Conjecture 6.5. *The area distribution of JT gravity in positive curvature coupled to $c \leq 0$ conformal matter, with critical exponent ν given by Eq. (4.57), has the asymptotic behaviour*

$$\rho^{+}_{\beta_q}(A) \underset{A\to+\infty}{\sim} \frac{\kappa^{+}_{\nu}(\beta_q^{\nu}/L)}{L^2}\left(\frac{L^2}{A}\right)^{\chi^{+}_{\nu}}, \tag{6.23}$$

where κ^{+}_{ν} is a strictly positive function of the dimensionless parameter β_q^{ν}/L and χ^{+}_{ν} a critical exponent, which is a strictly increasing function of ν satisfying $\lim_{\nu\to\frac{1}{2}+}\chi^{+}_{\nu}=2$ and $\lim_{\nu\to 1-}\chi^{+}_{\nu}=+\infty$.

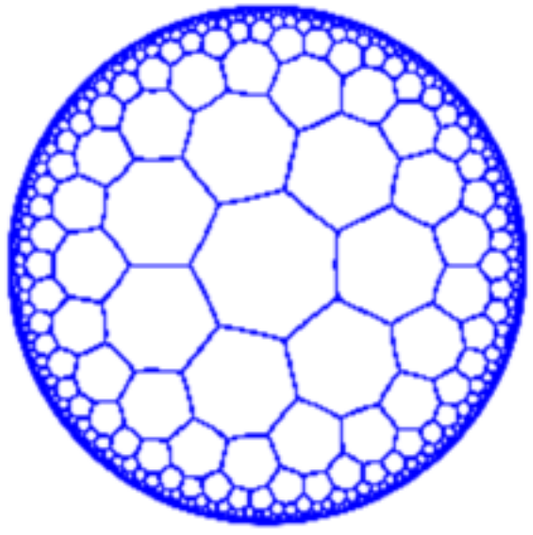
Emergence of the Schwarzian description and ballistic behaviour

The usual smooth length parameter ℓ used in the standard Schwarzian description is an effective macroscopic parameter. On short scales, the boundary is a fractal and it does not have a smooth length.

The smooth macroscopic length parameter ℓ can thus in principle be expressed in terms of the microscopic parameters.

The emergence of this parameter is related to the fundamental property of hyperbolic space, that the area grows exponentially with the geodesic radius.

In a lattice formulation, this is reflected by the fact that the number of lattice sites that are accessible at a given distance of a given site grows exponentially with the distance (only quadratically in flat space, or not at all in positive curvature).



Consider now a Brownian particule in hyperbolic space (an interesting simplified model). There will be two regimes.

$$\text{Small } \beta : d \sim \sqrt{\beta}$$

$$\text{Large } \beta : d \sim \beta / L$$

A smooth length emerges, with d proportional to the number of microscopic steps in the random motion.

In JT gravity, one can argue that $\ell \sim \beta \Big(|\Lambda| L \Big)^{\frac{1}{\nu} - 1}$

Continuum/probabilistic approach

Method: describe the metric by going to conformal gauge.

$$g = e^{2\Sigma}\delta$$

Puzzle: the constant curvature constraint imposes that

$$\Delta_\delta \Sigma = \frac{\eta}{L^2} e^{2\Sigma}$$

Imposing Dirichlet or Neumann boundary conditions would kill all the degrees of freedom. One must work with free boundary conditions.

A fundamental mathematical theorem ensures that, for a given boundary Liouville field, the bulk field is uniquely determined (in flat space, this is the usual statement about the uniqueness of harmonic functions on the disk, with prescribed value on the boundary; it also works in negative curvature; it is more subtle in positive curvature).

Fundamental degree of freedom: $\sigma = \Sigma_{|\partial\mathscr{D}}$

$$\Sigma = \Sigma_{\mathrm{B}} + \Sigma_{\sigma}$$

$$\Sigma_{\mathrm{B}|\partial\mathscr{D}} = 0$$

Σ_{σ} solves the Liouville equation is depends only on the boundary value σ

So “all” one needs to do is to construct a suitable (PSL(2,R)-invariant) probability measure on the space of boundary Liouville fields. This will automatically yield a measure on the space of constant curvature metrics or, equivalently, on the space of self-overlapping polygons!

Such measures are not unique.

But we can follow the usual path integral physics lore to construct one (the starting point is heuristic, the end point is rigorous).

The conjecture is that the measure obtained in this way matches with the one obtained by taking the continuum limit of the Self-Overlapping Polygon model (with the non-uniform measure).

This yields highly non-trivial predictions. Similar to the map counting/ Liouville theory relation in ordinary 2D quantum gravity.

We have tested these ideas semi-classically (with S. Chaudhuri).

PSL(2,R)-invariant measure (flat case)

$$D\sigma\, e^{-S}$$

$$S = \frac{1}{4\pi}\int_0^{2\pi}\Big(\sigma\,\mathrm{Hilb}[\sigma'] + 2\sigma\Big)\mathrm{d}\theta = \sum_{n\geq 1} n|\sigma_n|^2 + \sigma_0$$

This action predicts that

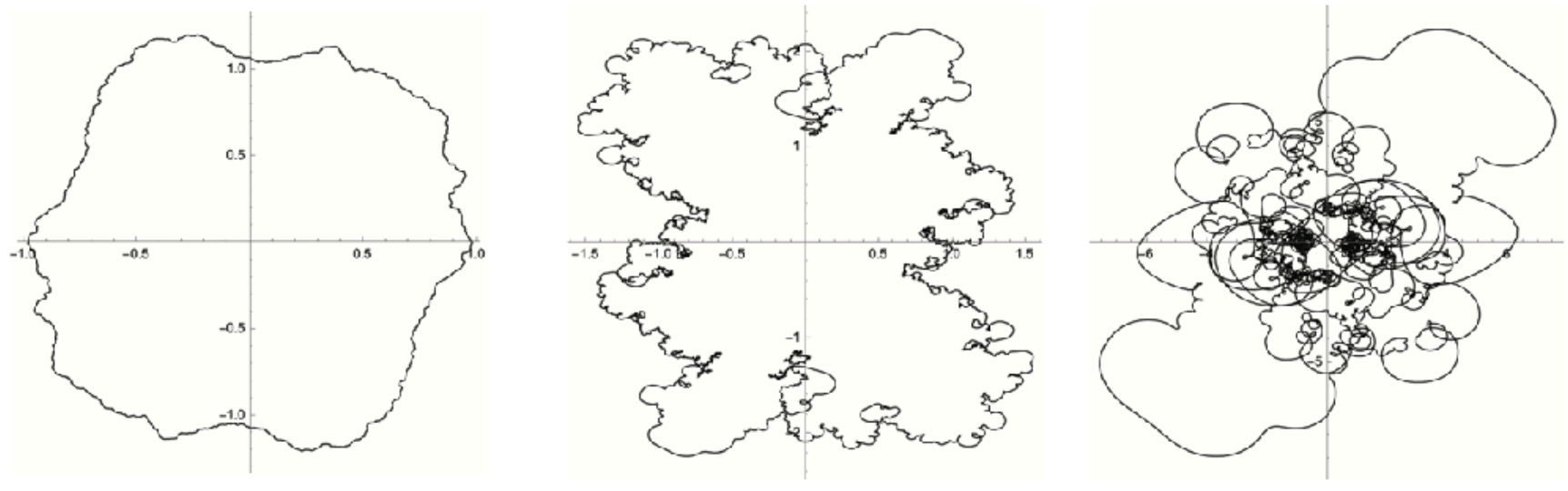
$$|\sigma_n| \sim \frac{1}{\sqrt{n}}$$

almost surely, implying that the boundary field is distribution-valued (very unlike a normal one-dimensional quantum variable q, which would have modes going like 1/n and would be continuous).

The bulk field is still perfectly smooth

$$\Sigma = \sum_{n\in\mathbb{Z}} \sigma_n \rho^{|n|} e^{in\theta} \quad \text{if} \quad \sigma = \sum_{n\in\mathbb{Z}} \sigma_n e^{in\theta}$$

This is how the pictures for typical configurations were obtained.



The boundary field is a one dimensional log-correlated scalar field. Well-studied by mathematicians and related to standard Liouville technology (vertex operators etc.). One can then use arguments à la DDK to derive

$$\nu = \frac{1}{2}\left[1 + \sqrt{\frac{c}{c-24}}\right]$$

One may also be able to adapt the strategies used in DOZZ, FZZT and mathematicians to derive exact results. Tbc.

Thank you for your attention !