

Continued fractions using a Laguerre digraph interpretation of the Foata–Zeilberger bijection and its variants

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- 1 Continued fractions and enumerative combinatorics
 - 1 Classical continued fractions
 - 2 Sokal–Zeng's results for factorials
 - 3 D.–Sokal's results for Genocchi and median Genocchi numbers
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 - 1 Flajolet's combinatorial interpretation
 - 2 Foata–Zeilberger bijection
- 3 What's new
 - 1 Laguerre digraphs
 - 2 New interpretation of the FZ bijection
- 4 The story continues . . .

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Continued fractions for ogf

Given sequence $(a_n)_{n \geq 0}$

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$$\sum_{n=0}^{\infty} a_n t^n = \frac{1}{1 - \frac{\alpha_1 t}{1 - \frac{\alpha_2 t}{1 - \ddots}}}$$

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Stieltjes-type continued fractions (S-fractions)

An example

$$\sum_{n=0} n!t^n = \frac{1}{1 - \frac{1 \cdot t}{1 - \frac{1 \cdot t}{1 - \frac{2 \cdot t}{1 - \frac{2 \cdot t}{1 - \frac{3 \cdot t}{1 - \frac{3 \cdot t}{1 - \dots}}}}}}}$$

This line of thought goes back to Euler

$$A = \frac{1}{1+x} \cdot \frac{1+x}{1+2x} \cdot \frac{1+2x}{1+3x} \cdot \frac{1+3x}{1+4x} \cdot \frac{1+4x}{1+5x} \cdot \frac{1+5x}{1+6x} \cdot \frac{1+6x}{1+7x} \cdot \text{etc.}$$

§. 22. Quemadmodum autem huiusmodi fractionum continuorum valor fit investigandus, alibi ostendi: Scilicet cum singulorum denominatorum partes integre sint unitates, soli numeratores in computum veniunt; fit ergo $x=1$, atque investigatio summe A sequenti modo instituetur:

$$A = \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}, \text{etc.}$$

num. 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, etc:

Fractiones nimirum hic exhibitae continuo propius ad verum valorem ipsius A accedunt, et quidem alternatim eo sunt maiores et minores; ita ut fit:

Tom. V. Non. Com. F f A >

Euler (1760)

Jacobi-type continued fractions

$$\frac{1}{1 - \gamma_0 t - \frac{\beta_1 t^2}{1 - \gamma_1 t - \frac{\beta_2 t^2}{1 - \ddots}}}$$

J-fractions in short

J-fraction for $n!$

$$\sum_{n=0}^{\infty} n!t^n = \frac{1}{1 - \gamma_0 t - \frac{\beta_1 t^2}{1 - \gamma_1 t - \frac{\beta_2 t^2}{1 - \ddots}}}$$

where

$$\begin{aligned}\gamma_n &= 2n + 1 \\ \beta_n &= n^2\end{aligned}$$

Start with

$$\sum_{n=0}^{\infty} n!t^n = \frac{1}{1 - \gamma_0 t - \frac{\beta_1 t^2}{1 - \gamma_1 t - \frac{\beta_2 t^2}{1 - \ddots}}}$$

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Instead consider

$$\begin{aligned}\gamma_0 &= w_0 \\ \gamma_n &= [x_2 + (n-1)u_2] + [y_2 + (n-1)v_2] + w_n \\ \beta_n &= [x_1 + (n-1)u_1][y_1 + (n-1)v_1]\end{aligned}$$

Sokal–Zeng's (2022) reverse program

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Question

Find the 10 permutation statistics $x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w_0, (w_n)_{n \geq 1}$

Continued fractions by Sokal–Zeng (2022) for permutations

“First” (Cycles not counted)

J-fraction with 10-statistics

“Second” (Cycles counted)

Conjecture: J-fraction with 10-statistics

Continued fractions by Sokal–Zeng (2022) for permutations

“First” (Cycles not counted)	“Second” (Cycles counted)
J-fraction with 10-statistics	Conjecture: J-fraction with 10-statistics ↓ J-fraction with 9-statistics

Continued fractions by Sokal–Zeng (2022) for permutations

“First” (Cycles not counted)	“Second” (Cycles counted)
<p data-bbox="83 370 529 405">J-fraction with 10-statistics</p> <p data-bbox="157 419 179 453">↑↑</p> <p data-bbox="83 458 406 493"><i>p, q</i>-generalisation:</p> <p data-bbox="83 503 513 537">J-fraction with 18 statistics</p>	<p data-bbox="665 280 1308 314">Conjecture: J-fraction with 10-statistics</p> <p data-bbox="739 329 761 363">⇓</p> <p data-bbox="665 368 1092 403">J-fraction with 9-statistics</p> <p data-bbox="739 417 761 451">↑↑</p> <p data-bbox="665 456 985 491"><i>p, q</i>-generalisation:</p> <p data-bbox="665 501 1094 535">J-fraction with 15 statistics</p>

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They later discovered that Randrianarivony (1998) had a 17-variable continued fraction.

Results and conjecture for factorials

Cycle classification

For a permutation σ , compare each i with $\sigma(i)$ and $\sigma^{-1}(i)$:

Cycle classification

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- cycle valley $\sigma^{-1}(i) > i < \sigma(i)$
- cycle peaks $\sigma^{-1}(i) < i > \sigma(i)$
- cycle double rise $\sigma^{-1}(i) < i < \sigma(i)$
- cycle double fall $\sigma^{-1}(i) > i > \sigma(i)$
- fixed point $i = \sigma(i) = \sigma^{-1}(i)$

Consider σ as $\sigma(1)\sigma(2)\dots\sigma(n)$:

Record classification

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- i is record if for every $j < i$ we have $\sigma(j) < \sigma(i)$
left-to-right-maxima
- i is antirecord if for every $j > i$ we have $\sigma(j) > \sigma(i)$
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Each i is one of the following four types:

- rar - record-antirecord
- errec - exclusive record
- earec - exclusive antirecord
- nrar - neither record-antirecord

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Each i is one of the following ten (not 20) types:

Record-and-cycle classification

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	cpeak	cval	cdrise	cdfall	fix
erec		ereccval	ereccdrise		
earec	eareccpeak			eareccdfall	
rar					rar
nrar	nrcpeak	nrcval	nrcdrise	nrcdfall	nrfix

Consider 10-variable polynomials

$$P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\text{eareccpeak}(\sigma)} x_2^{\text{eareccdfall}(\sigma)} y_1^{\text{ereccval}(\sigma)} y_2^{\text{ereccdrise}(\sigma)} z^{\text{rar}(\sigma)} \times \\ u_1^{\text{nrcpeak}(\sigma)} u_2^{\text{nrcdfall}(\sigma)} v_1^{\text{nrcval}(\sigma)} v_2^{\text{nrcdrise}(\sigma)} w^{\text{nrfix}(\sigma)}$$

Continued fractions counting permutation statistics

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Theorem (Sokal–Zeng (2022) First J-fraction for permutations)

$$\begin{aligned} & \sum_{n=0}^{\infty} P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) t^n \\ = & \frac{1}{1 - z \cdot t - \frac{1}{x_1 y_1 \cdot t^2} \frac{1}{(x_1 + u_1)(y_1 + v_1) \cdot t^2} \frac{1}{1 - ((x_2 + u_2) + (y_2 + v_2) + w) \cdot t - \frac{(x_1 + 2u_1)(y_1 + 2v_1) \cdot t^2}{1 - \dots}}} \end{aligned}$$

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Proof uses the Foata–Zeilberger bijection (1990)

Can we count cycles as well?

Consider 11-variable polynomials

$$P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\text{eareccpeak}(\sigma)} x_2^{\text{eareccdfall}(\sigma)} y_1^{\text{ereccval}(\sigma)} y_2^{\text{ereccdrise}(\sigma)} z^{\text{rar}(\sigma)} \times \\ u_1^{\text{nrcpeak}(\sigma)} u_2^{\text{nrcdfall}(\sigma)} v_1^{\text{nrcval}(\sigma)} v_2^{\text{nrcdrise}(\sigma)} w^{\text{nrfix}(\sigma)} \lambda^{\text{cyc}(\sigma)}$$

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Conjecture (Sokal–Zeng (2022))

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, y_1, y_2, u_1, u_2, y_1, v_2, w, z, \lambda) t^n \\ = \frac{1}{1 - \lambda z \cdot t - \frac{\lambda x_1 y_1 \cdot t^2}{1 - (x_2 + y_2 + \lambda w) \cdot t - \frac{(\lambda + 1)(x_1 + u_1) y_1 \cdot t^2}{1 - ((x_2 + v_2) + (y_2 + v_2) + \lambda w) \cdot t - \frac{(\lambda + 2)(x_1 + 2u_1) y_1 \cdot t^2}{1 - \dots}}}}$$

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Genocchi and median Genocchi numbers

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The first few numbers are 1, 1, 3, 17, 155, 2073, ...

Genocchi numbers:

D-e-semiderangements

$$g_n = \#\{\sigma \in \mathfrak{S}_{2n} \mid 2i > \sigma(2i) \text{ and } 2i - 1 \leq \sigma(2i - 1)\}$$

D-o-semiderangements

$$= \#\{\sigma \in \mathfrak{S}_{2n} \mid 2i \geq \sigma(2i) \text{ and } 2i - 1 < \sigma(2i - 1)\}$$

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Median Genocchi numbers:

D-permutations (introduced by Lazar and Wachs (2019))

$$h_{n+1} = |\mathcal{D}_{2n}| = \#\{\sigma \in \mathfrak{S}_{2n} \mid 2i \geq \sigma(2i) \text{ and } 2i - 1 \leq \sigma(2i - 1)\}$$

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D — Dumont-like

Continued fractions for median Genocchi numbers

The once-shifted median Genocchi numbers h_{n+1} have the following Thron-type continued fraction

$$\sum_{n=0}^{\infty} h_{n+1} t^n = \frac{1}{1 - t - \frac{1t}{1 - \frac{4t}{1 - \frac{4t}{1 - \frac{9t}{1 - \dots}}}}}$$

Story for Genocchi and median Genocchi

D.-Sokal (2024):

Story for Genocchi and median Genocchi

D.–Sokal (2024):

“First” (Cycles not counted)	“Second” (Cycles counted)
<p>0-T-fraction with 12-statistics ↑↑ p, q-generalisation: 0-T-fraction with 22 statistics ↑↑ Master T-fraction: four infinite 2-parameter families two infinite 1-parameter family</p> <p>Proof: FZ-like bijection</p>	<p>Conjecture: 0-T-fraction with 12-statistics</p> <p>0-T-fraction with 12-statistics ↑↑ p, q-generalisation: 0-T-fraction with 21 statistics ↑↑ Master T-fraction: three infinite 2-parameter families, three infinite 1-parameter families, and one statistic for counting cycles</p> <p>Proof: Biane-like bijection</p>
Variant forms with slightly different statistics	

Randrianarivony–Zeng conjecture (1996)

Define the polynomials

where
$$G_n(x, y, \bar{x}, \bar{y}) = \sum_{\sigma \in \mathfrak{D}_{2n}^{\circ}} x^{\text{comi}(\sigma)} y^{\text{lema}(\sigma)} \bar{x}^{\text{cemi}(\sigma)} \bar{y}^{\text{remi}(\sigma)}$$

Randrianarivony–Zeng conjecture (1996)

Define the polynomials

$$G_n(x, y, \bar{x}, \bar{y}) = \sum_{\sigma \in \mathfrak{D}_{2n}^o} x^{\text{comi}(\sigma)} y^{\text{lema}(\sigma)} \bar{x}^{\text{cemi}(\sigma)} \bar{y}^{\text{remi}(\sigma)}$$

where

- comi – odd cycle minima,
- lema – left-to-right maxima whose value is even
- cemi – even cycle minima
- remi – right-to-left minima whose value is even

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Conjecture

The ogf of the polynomials $G_n(x, y, \bar{x}, \bar{y})$ has the following S -fraction

$$\sum_{n=0}^{\infty} G_n(x, y, \bar{x}, \bar{y}) t^n = \frac{1}{1 - \frac{xyt}{1 - \frac{1(\bar{x} + \bar{y})t}{1 - \frac{(x+1)(y+1)t}{1 - \frac{2(\bar{x} + \bar{y} + 1)t}{1 - \frac{(x+2)(y+2)t}{1 - \frac{3(\bar{x} + \bar{y} + 2)t}{1 - \dots}}}}}} \quad (1)$$

Plot twist for Sokal–Zeng conjecture

“First” (Cycles not counted)

J-fraction with 10 statistics



p, q -generalisation:

J-fraction with 18 statistics



Master J-fraction:

four infinite 2-parameter families
one infinite 1-parameter family



Proof:

Foata–Zeilberger bijection (1990)

“Second” (Cycles counted)

Conjecture: J-fraction with 10 statistics



J-fraction with 9 statistics



p, q -generalisation:

J-fraction with 15 statistics



Master J-fraction:

three infinite 2-parameter families
two infinite 1-parameter family
and one statistic for counting cycles



Proof:

Biane bijection (1993)

Conjectures of D.–Sokal, Randrianarivony–Zeng

“First” (Cycles not counted)	“Second” (Cycles counted)
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<p>Variant forms</p>	<p>Variant forms \implies Randrianarivony–Zeng (1996) S-fraction with 4 statistics</p>

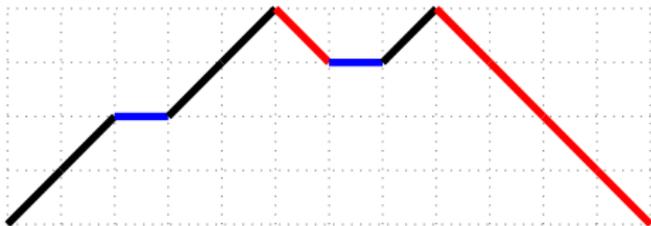
Pan–Zeng ('23) came up with multivariate continued fractions for other objects enumerated by Genocchi numbers also introduced in work of Lazar's PhD thesis

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Consider a Motzkin path, let's say

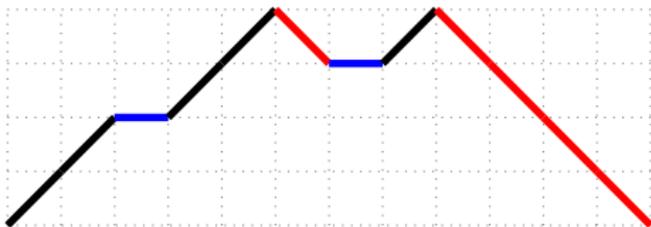
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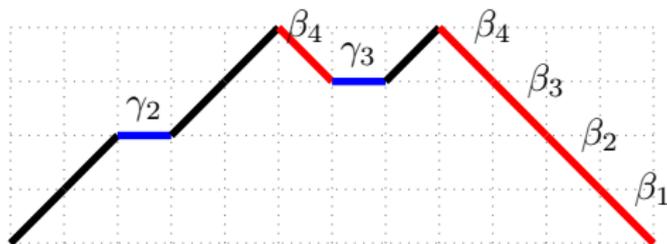


Assign weights:

- $\nearrow : 1$
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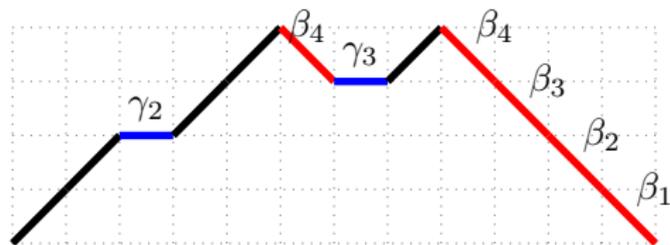


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$$\text{Weight} = \beta_1 \beta_2 \beta_3 \beta_4^2 \gamma_2 \gamma_3$$

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Combinatorial Interpretation of J-fraction

J-fraction

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The a_n are weighted sum of Motzkin paths with n steps.

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Gateway for proving continued fractions using bijective combinatorics :-D

Foata–Zeilberger bijection

excedance indices $F = \{i \in \sigma : \sigma(i) > i\} = \text{Cdrise} \cup \text{Cval}$

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Description of $\sigma \rightarrow \omega$

- If i is a cycle valley, step i is ↗
- If i is a cycle peak, step i is ↘
- If i is a cycle double rise, cycle double fall or fixed, step i is →, → or → respectively.

Description of labels $\sigma \rightarrow \xi$

For $i \in [n]$

$$\xi_i = \begin{cases} \#\{j: j < i \text{ and } \sigma(j) > \sigma(i)\} & \text{if } \sigma(i) > i & \text{if } i \in \text{Cval} \cup \text{Cdrise} \\ \#\{j: j > i \text{ and } \sigma(j) < \sigma(i)\} & \text{if } \sigma(i) < i & \text{if } i \in \text{Cpeak} \cup \text{Cdfall} \\ 0 & \text{if } \sigma(i) = i & \text{if } i \in \text{Fix} \end{cases}$$

An example

Let $\sigma = 715492638 = (1762)(3598)(4) \in \mathfrak{S}_9$.

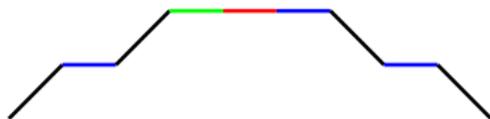
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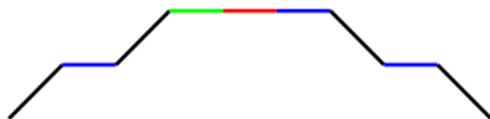


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The labels ξ and the sets F, F', G, G' are:

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Laguerre digraph

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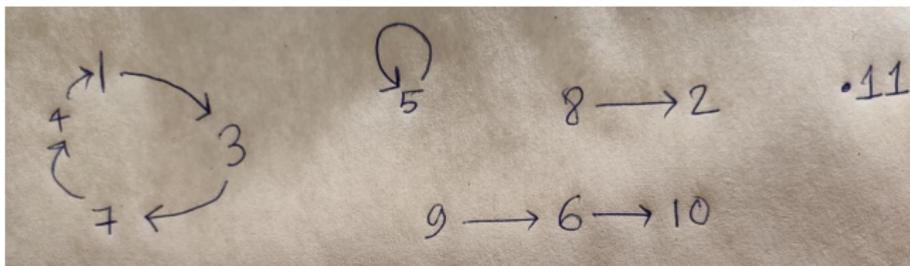
A **Laguerre digraph** of size n is a directed graph on vertex set $\{1, \dots, n\}$ where each vertex has indegree 0 or 1 and outdegree 0 or 1.

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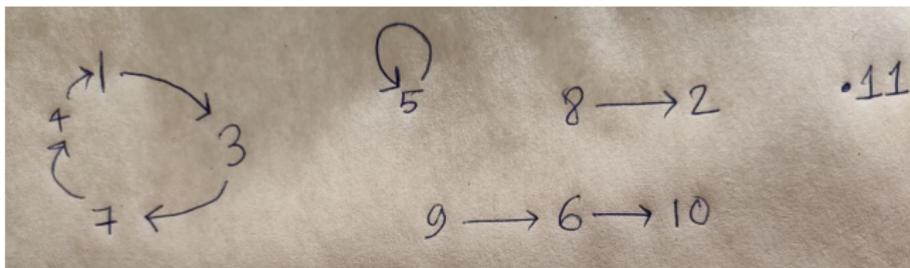


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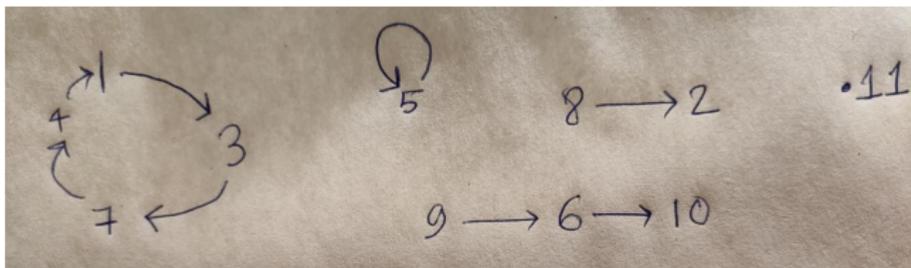
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Generalise permutations

An equivalent object with a different definition was introduced by Foata–Strehl (1984) — Laguerre configurations

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Laguerre digraphs after Sokal (2022)

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Start with all n vertices and no edges

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Stage (a): $i \in H$ (fixed points) in increasing order

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This order is suggested by the inverse bijection and the inversion tables

History with an example

Let $\sigma = 715492638 = (1762)(3598)(4) \in \mathfrak{S}_9$.

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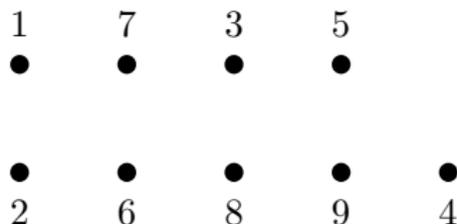
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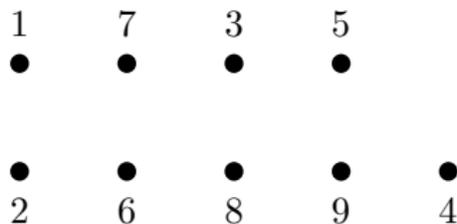
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Stage (a): H in increasing order



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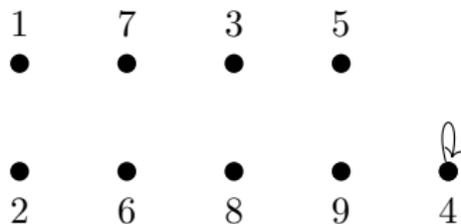
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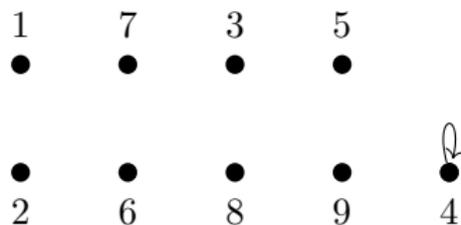
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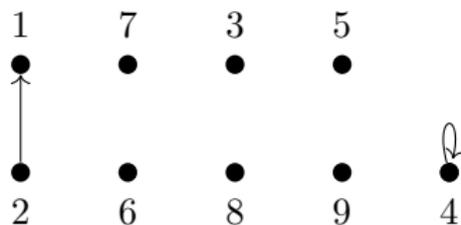
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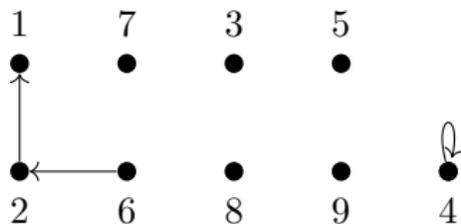
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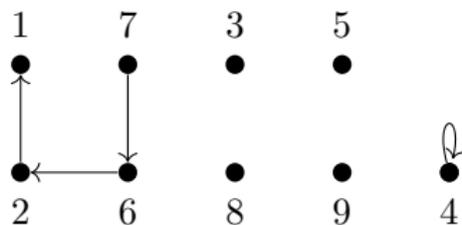
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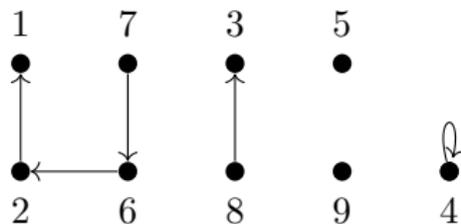
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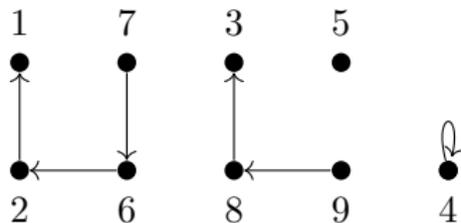
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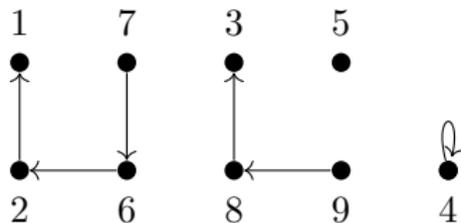
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$\sigma(i) \in G'$	1	2	6	3	8
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Stage (c): F in decreasing order



History with an example

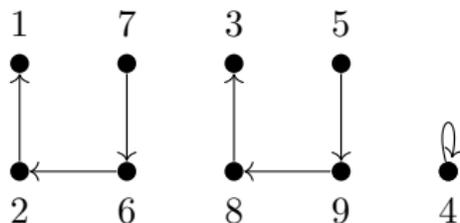
Let $\sigma = 715492638 = (1762)(3598)(4) \in \mathfrak{S}_9$.

$$H = \{4\}$$

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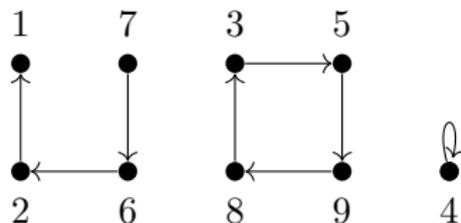
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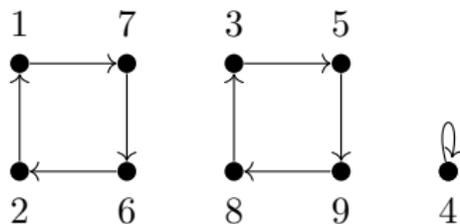
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Stage (c): F in decreasing order



- 1 Continued fractions and enumerative combinatorics
 - 1 Classical continued fractions
 - 2 Sokal–Zeng’s results for factorials
 - 3 D.–Sokal’s results for Genocchi and median Genocchi numbers
 - 4 Conjectures
- 2 Proof overview of existing results
 - 1 Flajolet’s combinatorial interpretation
 - 2 Foata–Zeilberger bijection
- 3 What’s new
 - 1 Laguerre digraphs
 - 2 New interpretation of the FZ bijection
- 4 The story continues . . .

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So there must be non-commutative analogues of continued fractions coming from Biane bijection or Françon–Viennot bijection (partially done by Kuba–Varvak '21)

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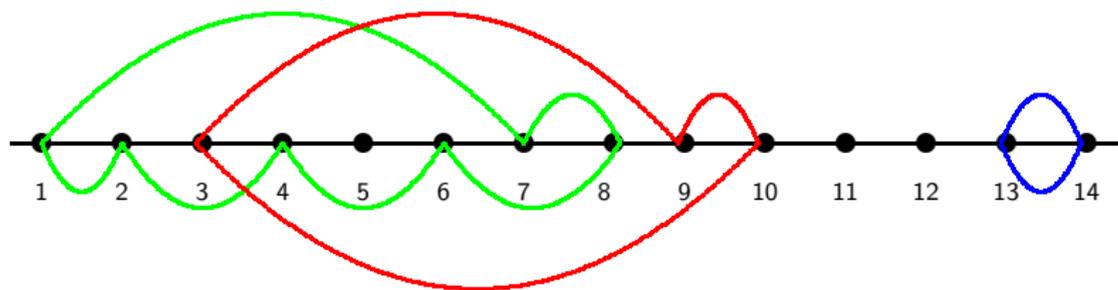
An index i is a descent of type 2 if i is a descent and i is a record (left-to-right maxima)

Merci pour votre attention

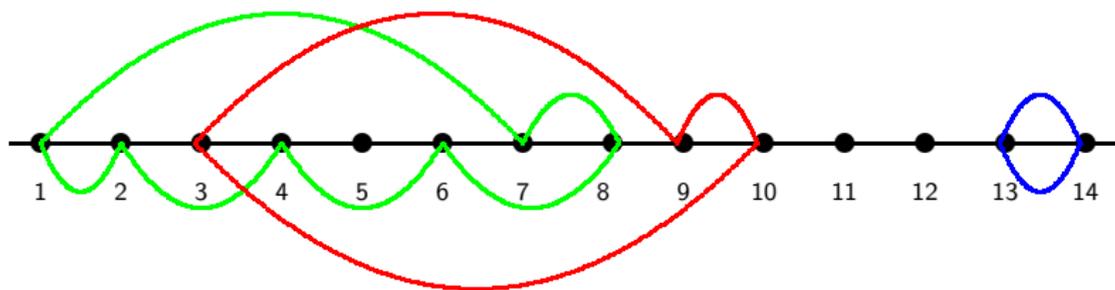


Sokal–Zeng's master J-fraction

Pictorial representation



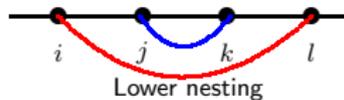
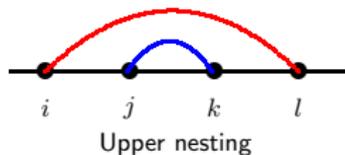
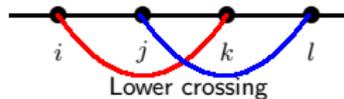
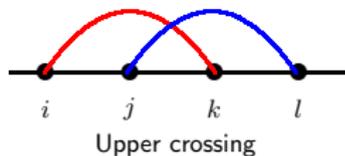
Pictorial representation



$$\sigma = 7192548610311121413 = (1, 7, 8, 6, 4, 2)(3, 9, 10)(5)(11)(12)(13, 14) \in \mathfrak{S}_{14}.$$

Due to Corteel (2007)

Crossings, nestings and pseudo-nestings

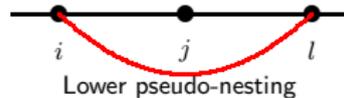
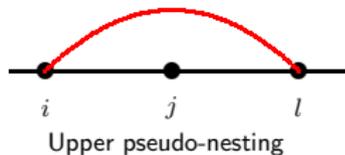


$\text{ucross}(j, \sigma)$

$\text{lcross}(k, \sigma)$

$\text{unest}(j, \sigma)$

$\text{lnest}(k, \sigma)$



Sokal–Zeng first master J-fraction for permutations

$$Q_n(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) =$$

$$\sum_{\sigma \in \mathfrak{S}_n} \prod_{i \in C_{\text{val}}(\sigma)} a_{\text{ucross}(i, \sigma), \text{unest}(i, \sigma)} \prod_{i \in C_{\text{peak}}(\sigma)} b_{\text{lcross}(i, \sigma), \text{lnest}(i, \sigma)} \times \\ \prod_{i \in C_{\text{dfall}}(\sigma)} c_{\text{lcross}(i, \sigma), \text{lnest}(i, \sigma)} \prod_{i \in C_{\text{drise}}(\sigma)} d_{\text{ucross}(i, \sigma), \text{unest}(i, \sigma)} \times \\ \prod_{i \in \text{Fix}(\sigma)} e_{\text{psnest}(i, \sigma)}$$

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$$\gamma_k = \left(\sum_{\xi=0}^{k-1} c_{k-1-\xi, \xi} \right) + \left(\sum_{\xi=0}^{k-1} d_{k-1-\xi, \xi} \right) + e_k$$

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(D. arxiv '23) master J-fraction for permutations

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