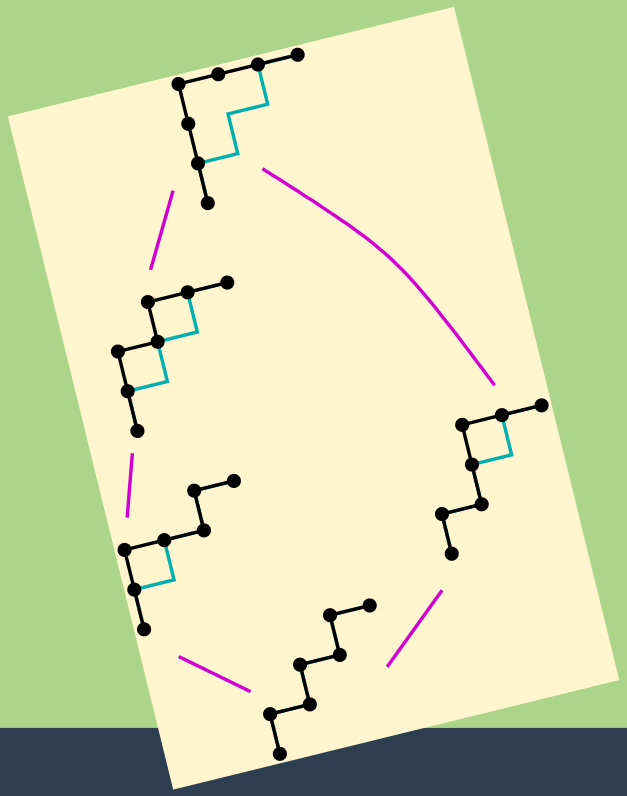
The ascent lattice on Dyck paths



with Jean-Luc Baril, Sergey Kirgizov
(LIB, Dijon) and Mehdi Naima (LIP6, Paris)

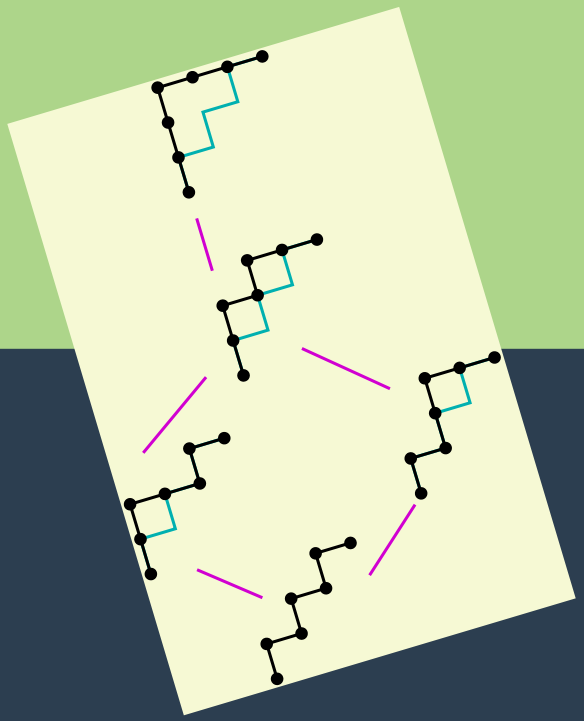
arXiv:2409.15982

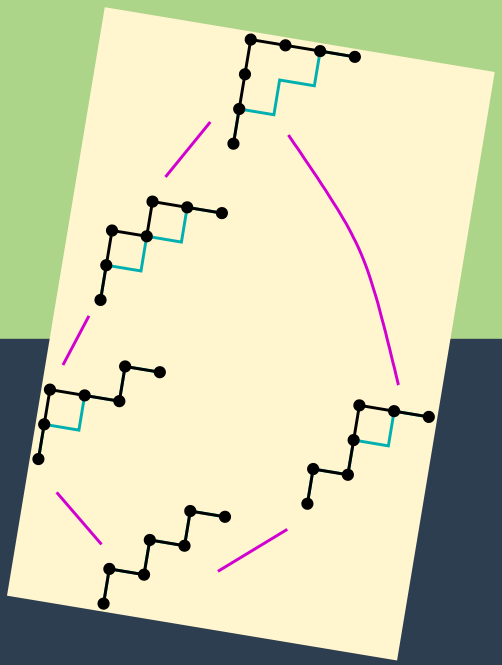
Mireille Bousquet-Mélou
CNRS, LaBRI, Université de Bordeaux

Outline

- Orders on Dyck paths --- The **ascent** order
- Lattice properties (the Nadeau-Tewari lattice)
- Counting intervals
- m-Dyck paths, and mirrored m-Dyck paths
- Connection with the sylvester congruence [Hivert, Novelli, Thibon 05]

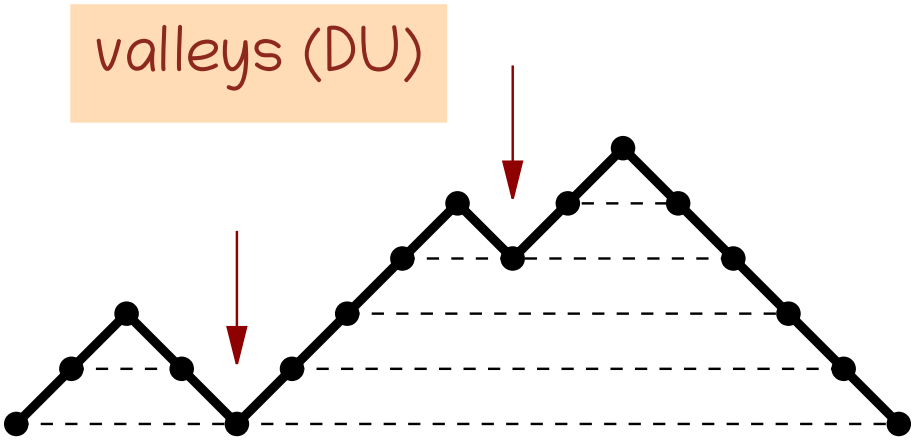
I. Orders on Dyck paths





Dyck paths

- A Dyck path of size n=8 (size=number of up steps)



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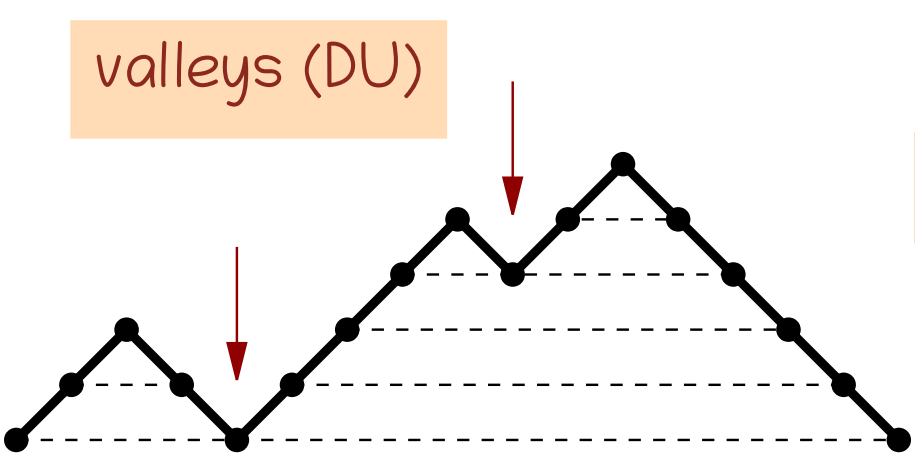
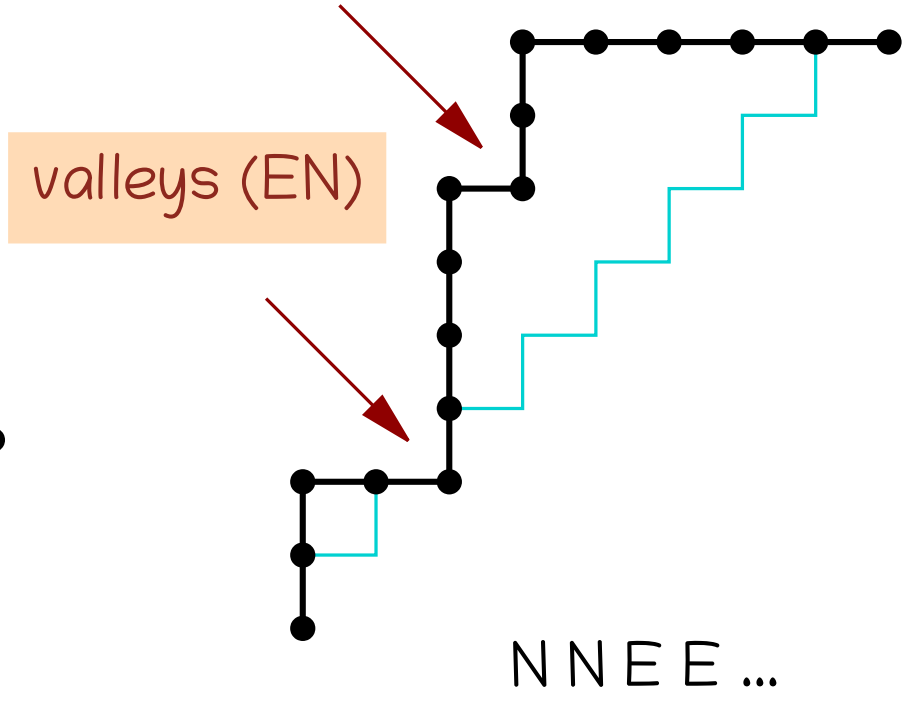
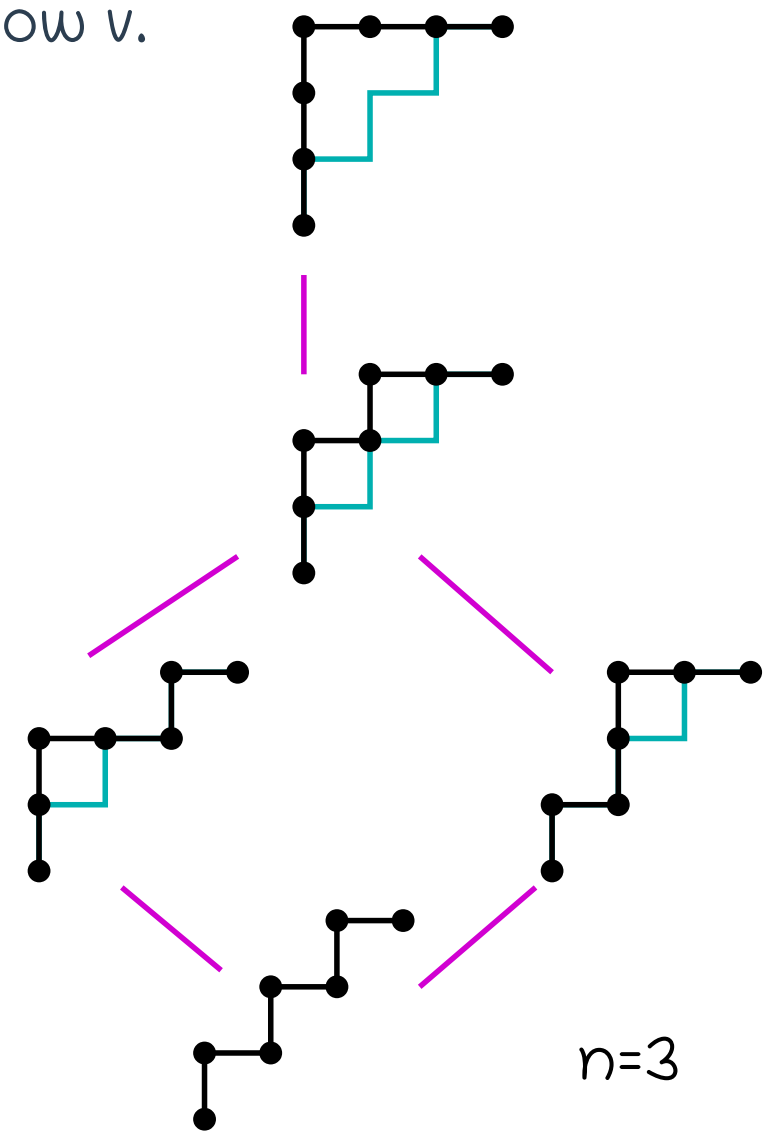




Stanley's lattice S_n

A poset on Dyck paths of size n

Def. The path u is smaller than v if u lies below v.



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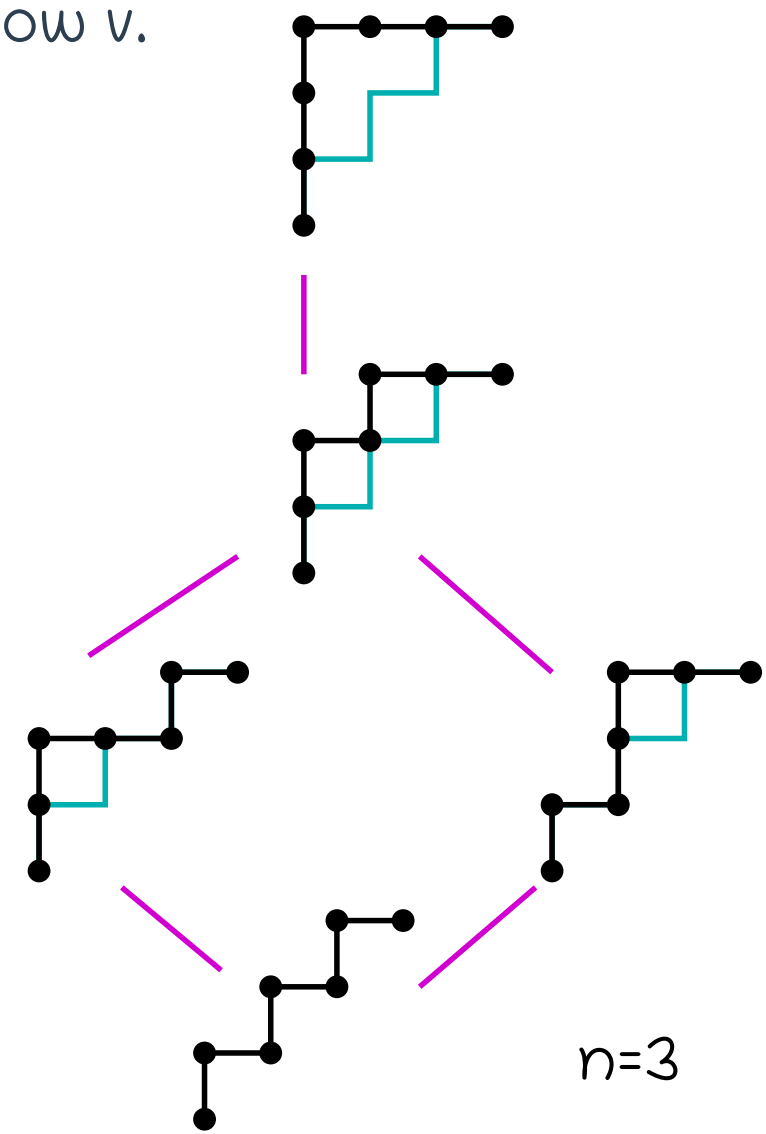
- Cover relations are given by

EN → NE (that is, valley → peak)



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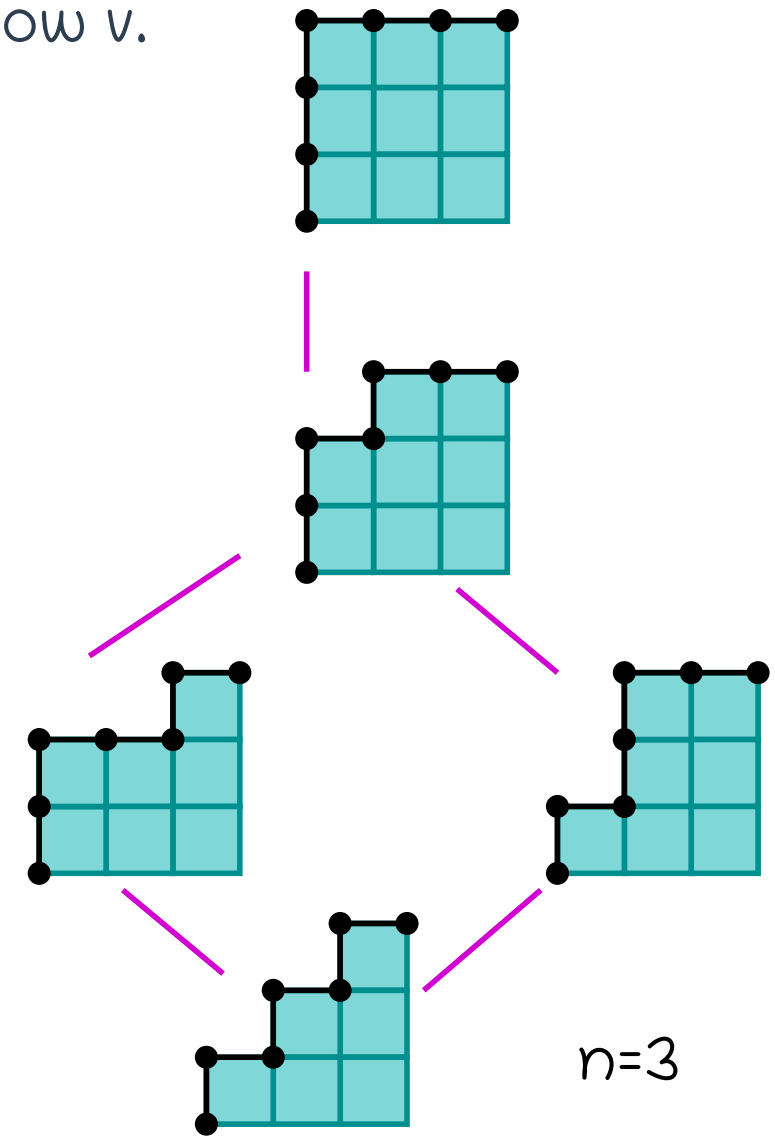
- A sub-lattice of the Young lattice



n=3

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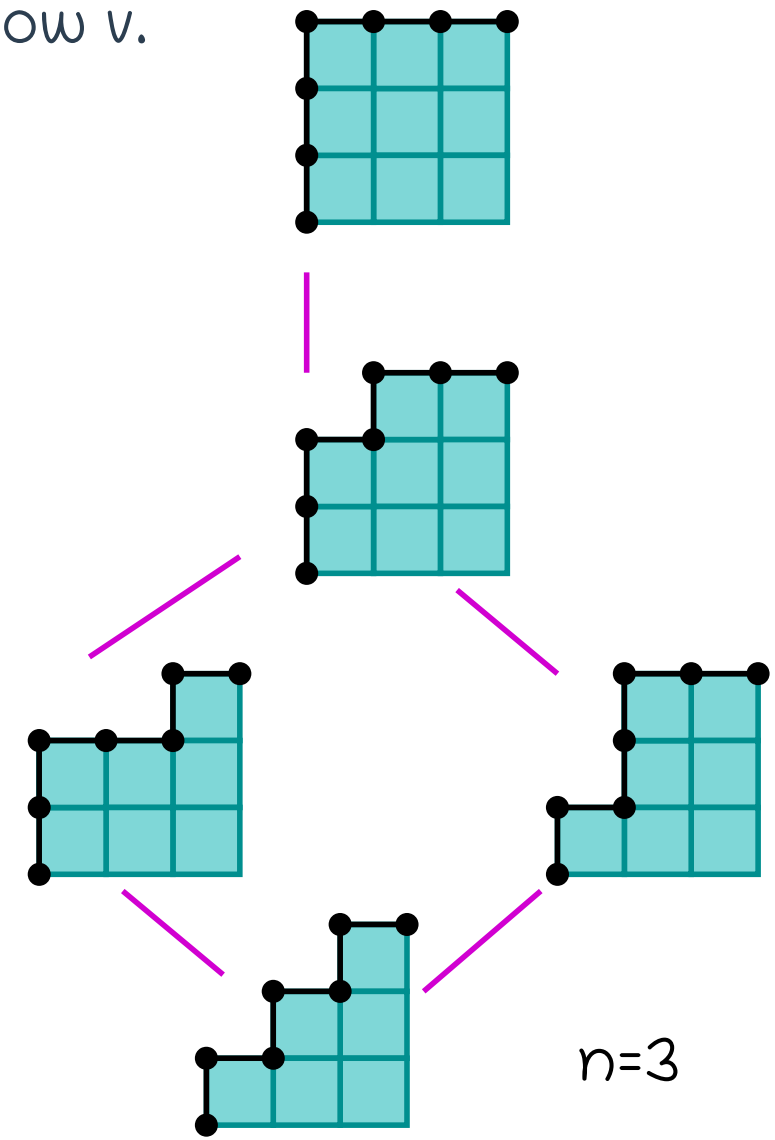
- Graded (by area)



n=3

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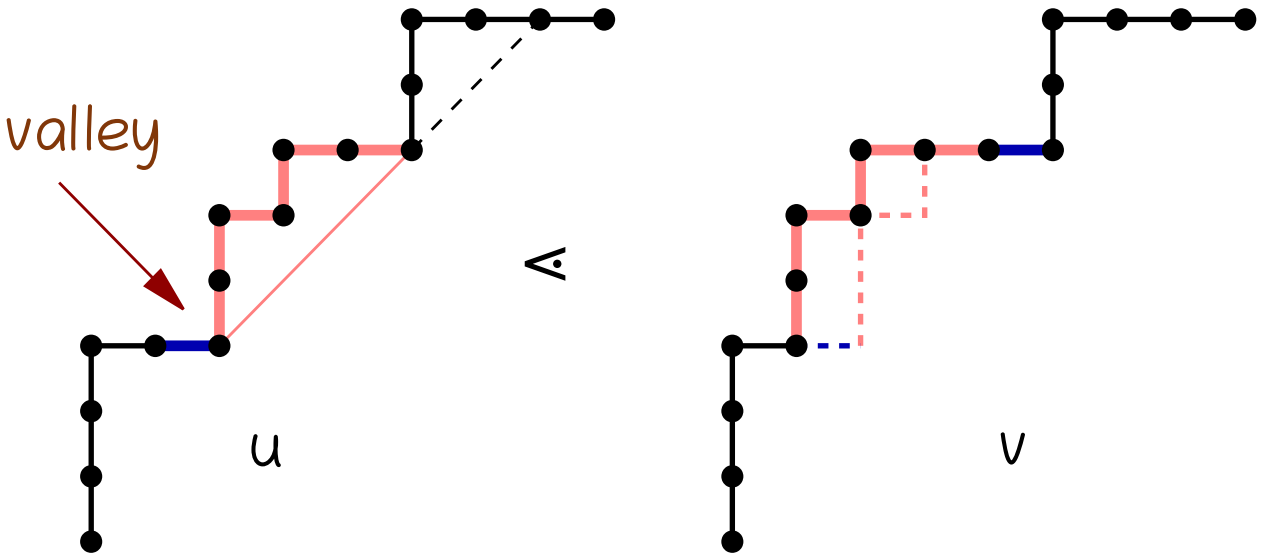
The Tamari lattice [Tamari 51]

- A poset on Dyck paths with n up steps
- Cover relations: choose a valley in the path u.

  Swap the East step and the shortest Dyck path that follows.



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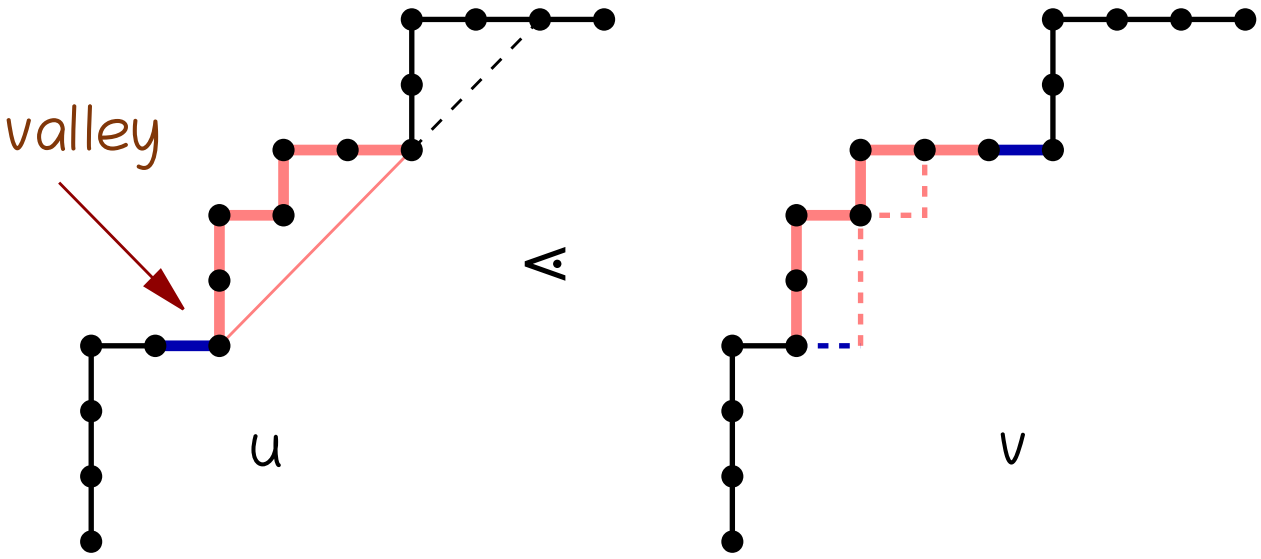

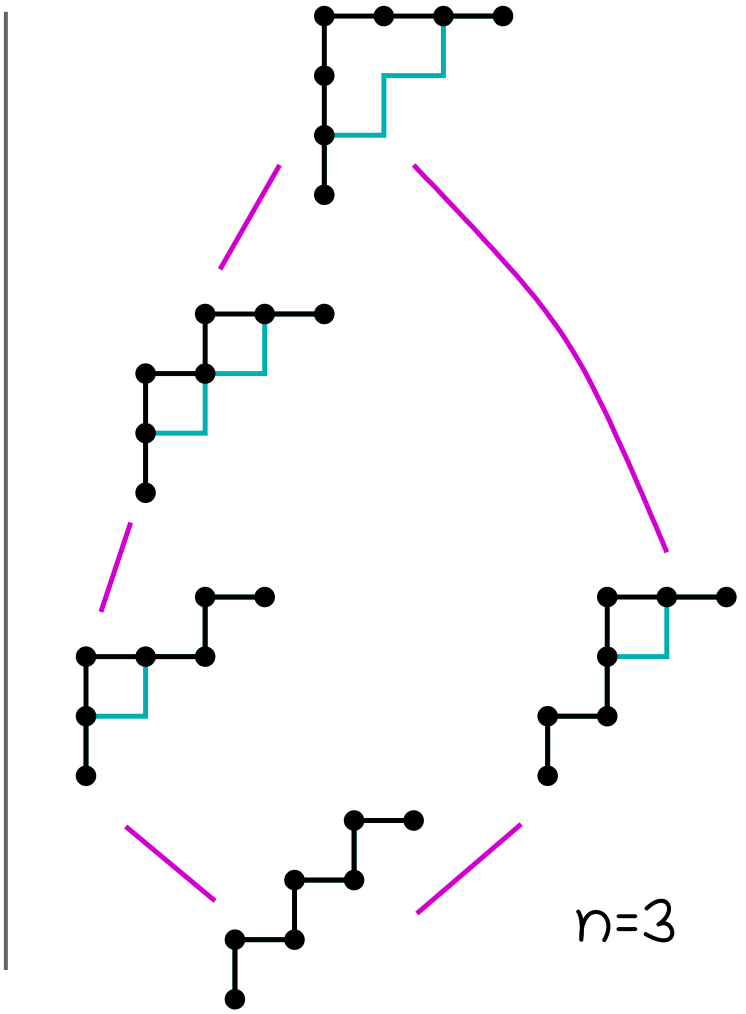

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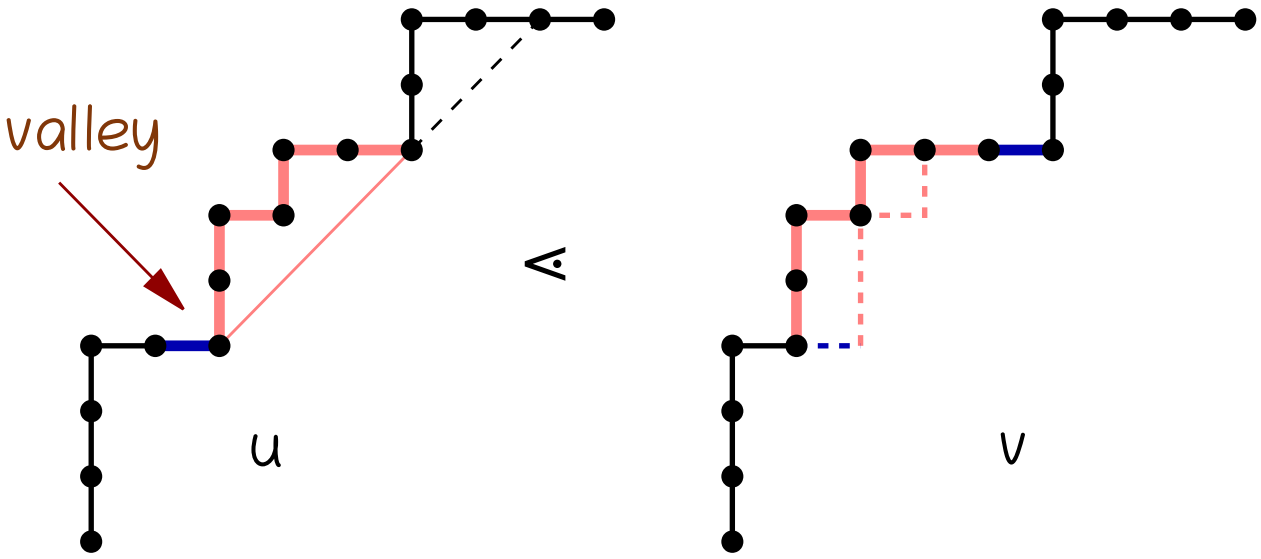

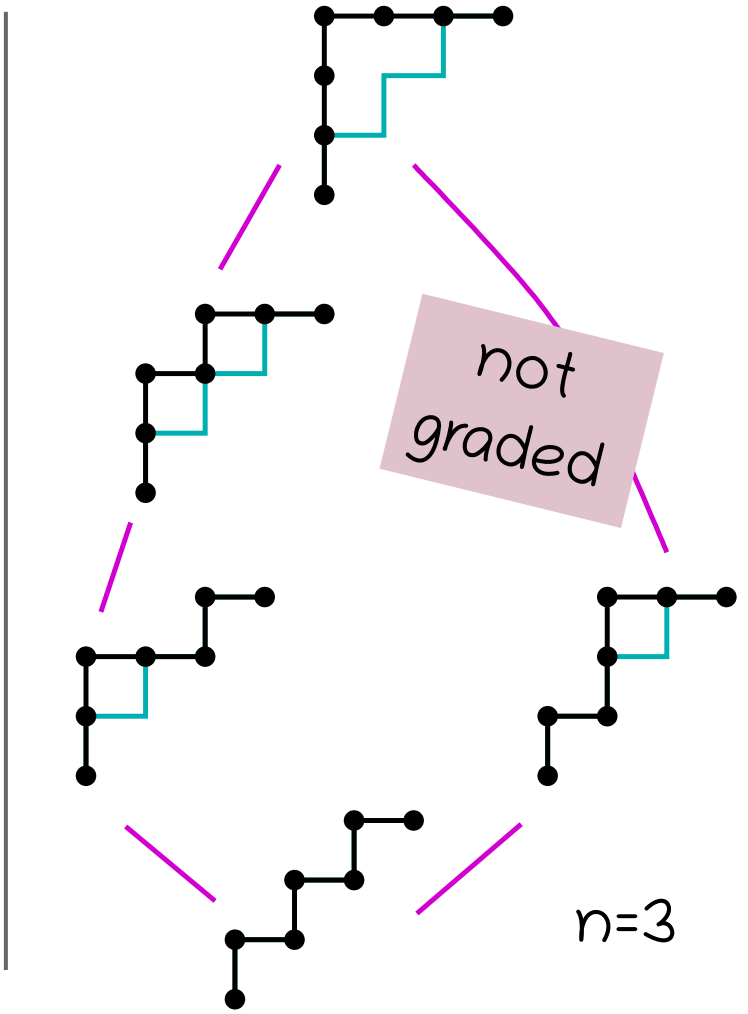

The Tamari lattice [Tamari 51]

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- Cover relations: choose a valley in the path u.

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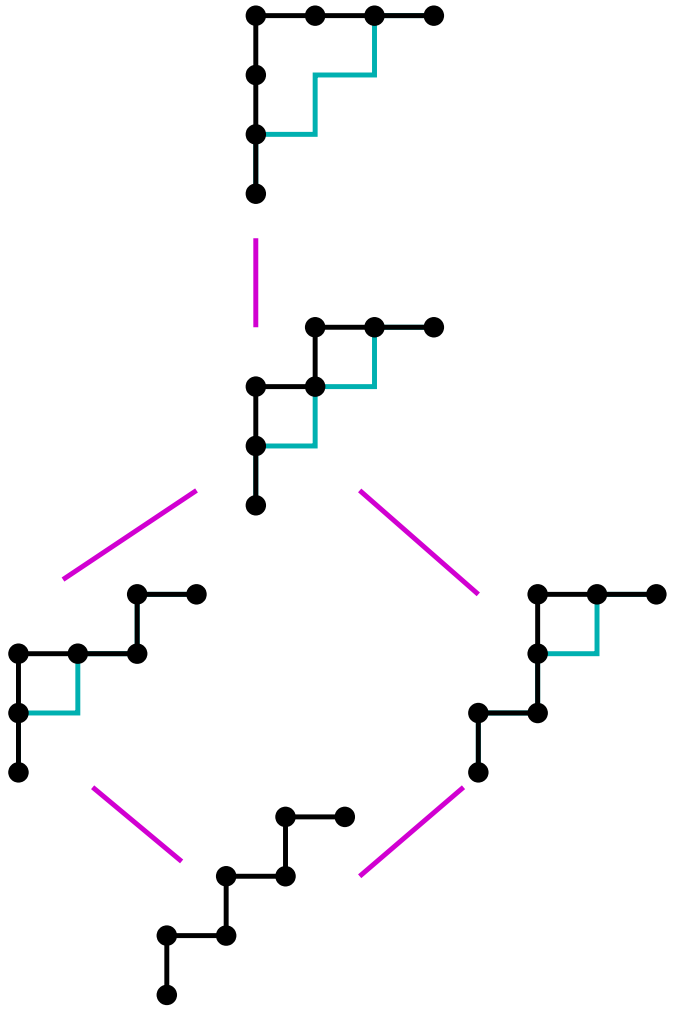
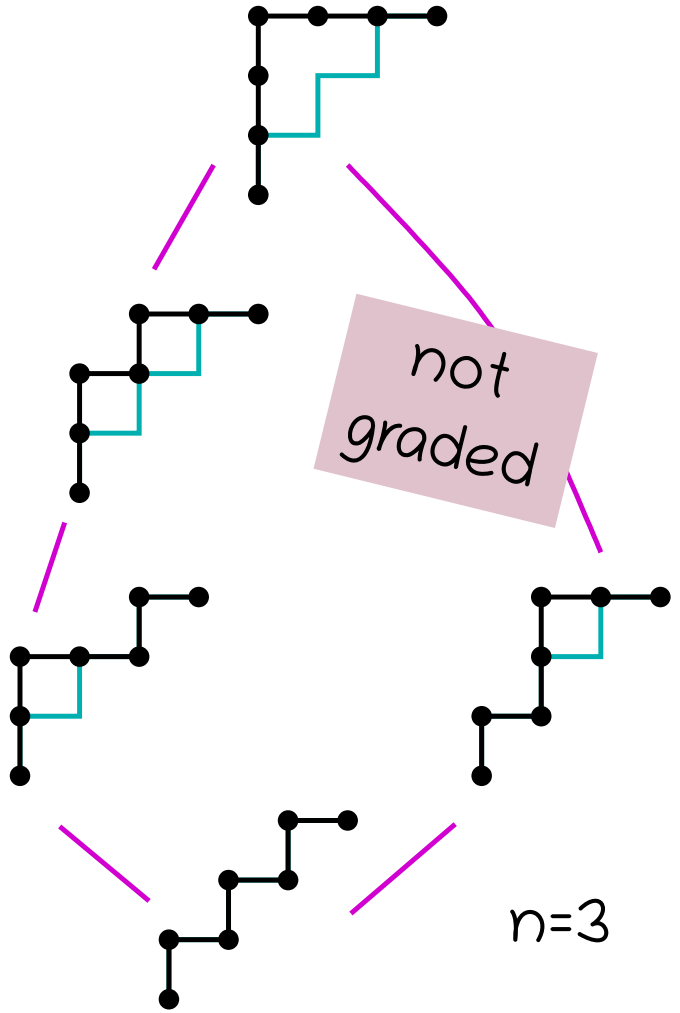
Stanley



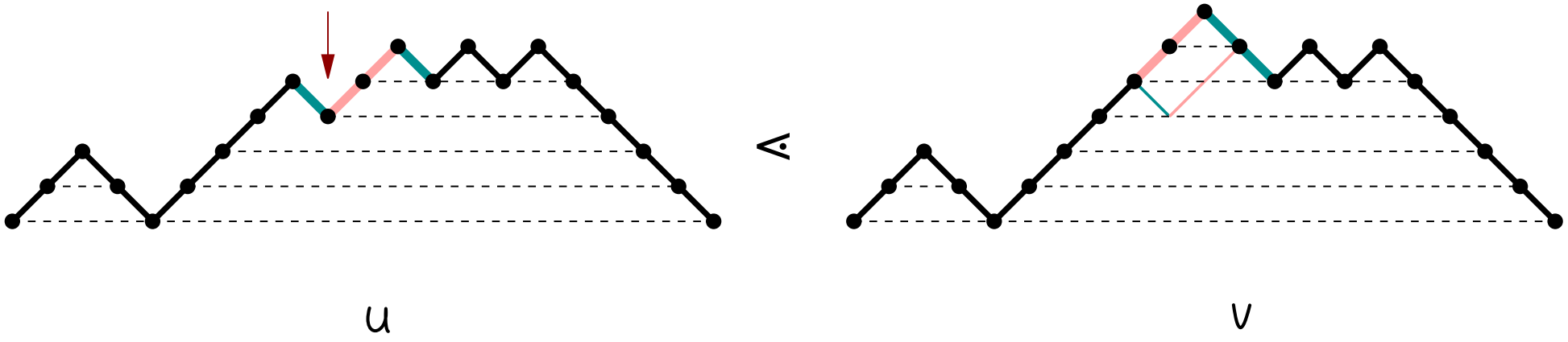


The ascent poset (or: greedy Stanley?)

- A poset on Dyck paths with n up steps
- Cover relations: choose a valley in the path u.

  Swap the down step and the ascent that follows.

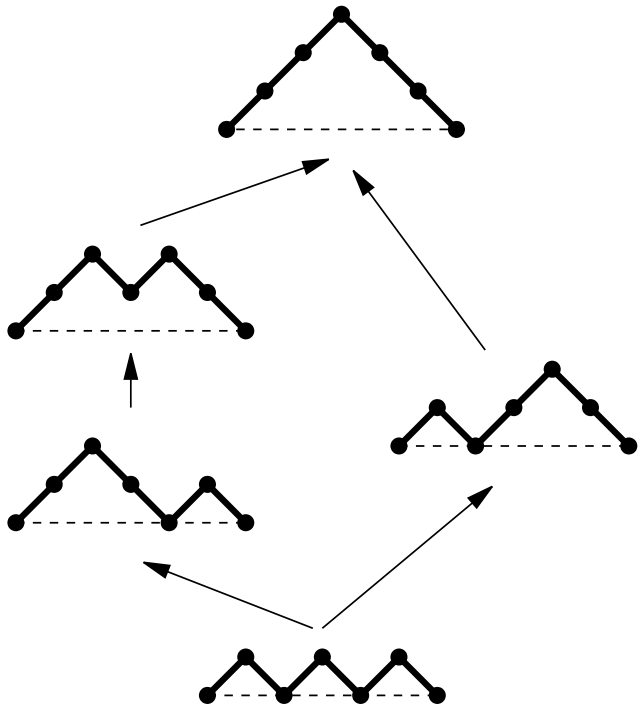
  (the path moves up)

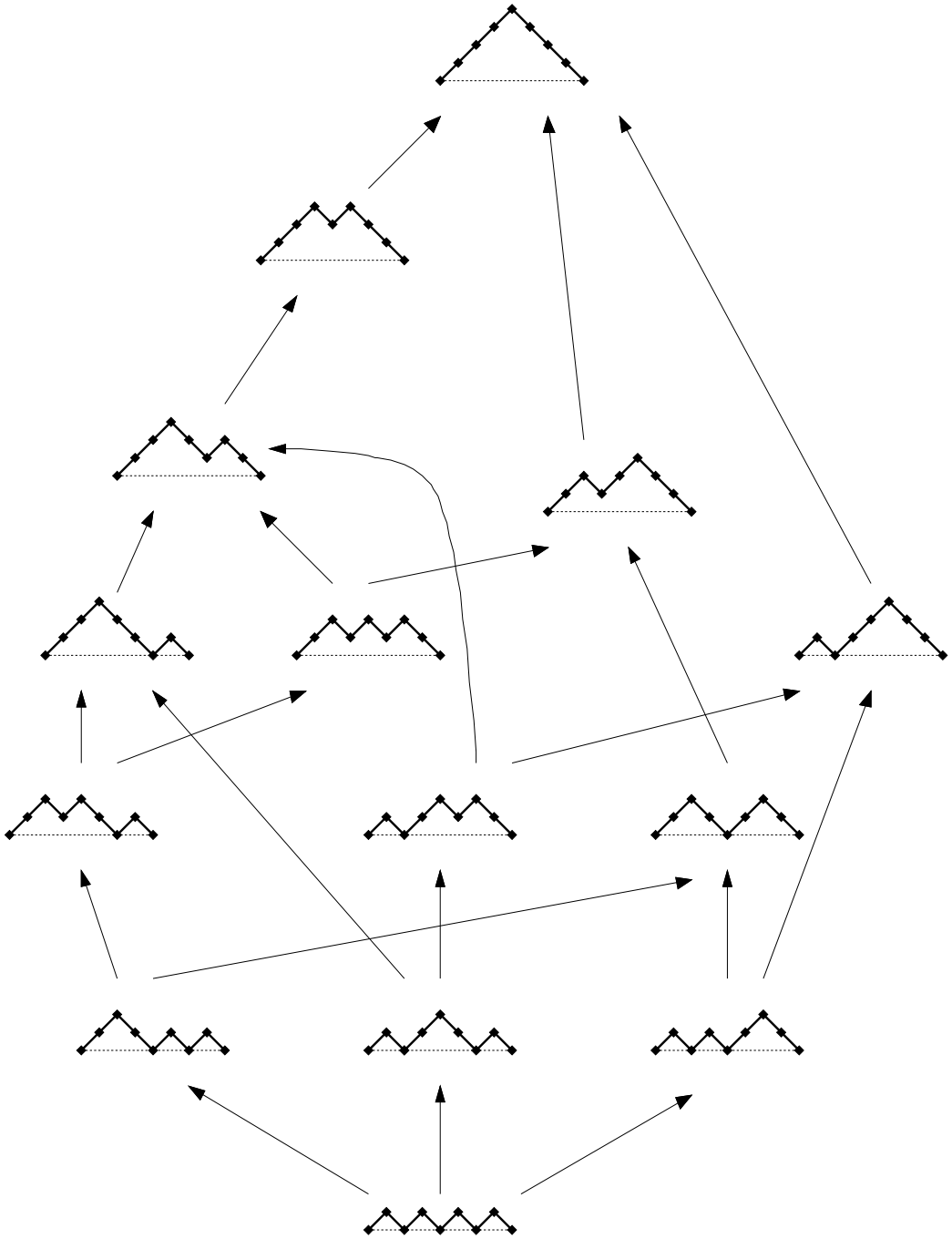


u $\lessdot$ v

[Chenevière, Nadeau...]

Ascent posets: n = 3, 4





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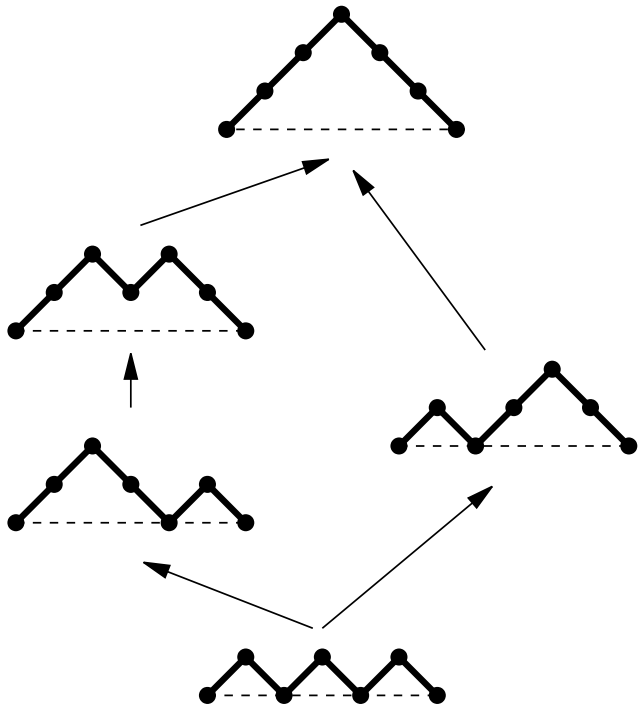
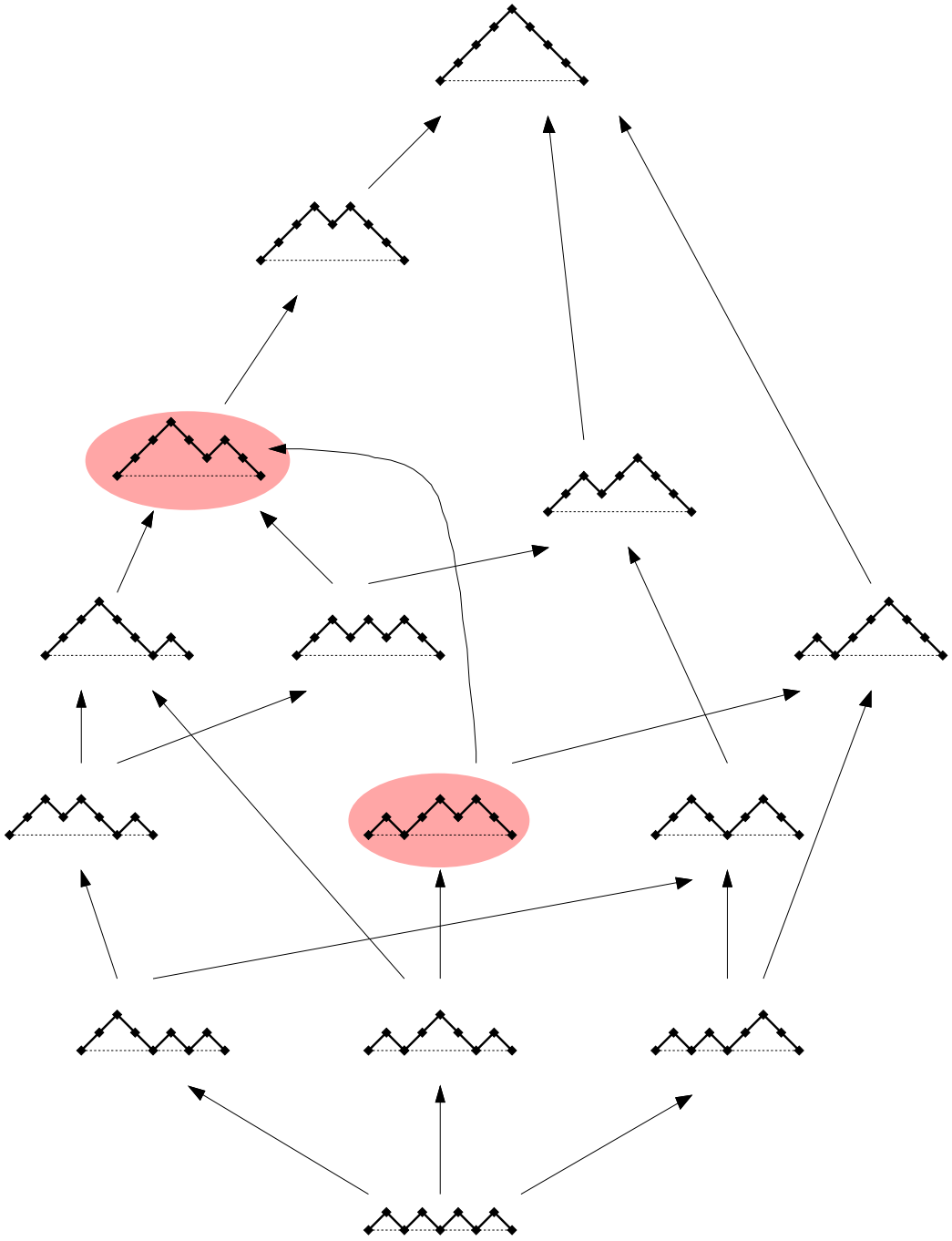
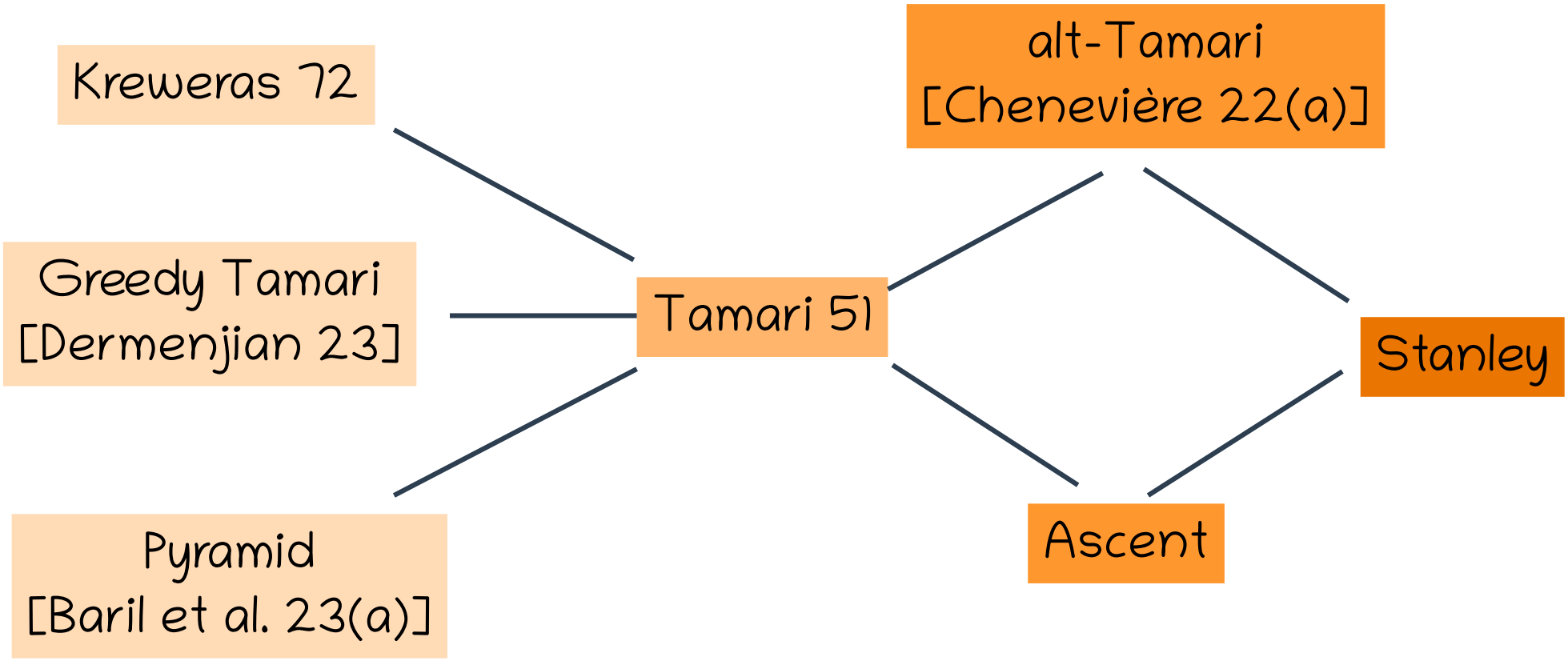




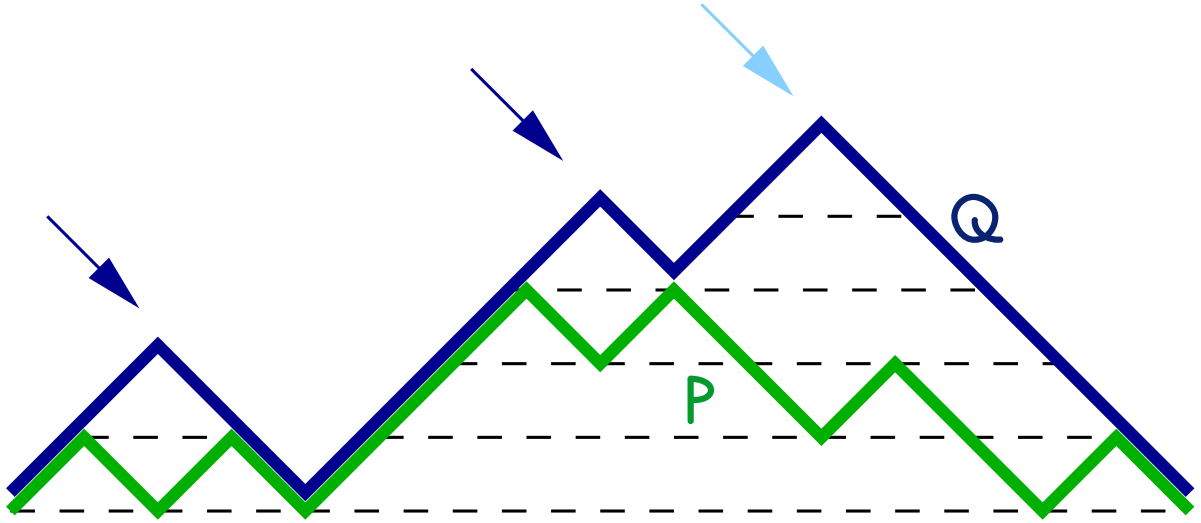
Poset inclusion



A characterization of the ascent order

Proposition. In the ascent poset, $P \preccurlyeq Q$ iff

- P lies below Q
- every descent of Q **is** (i.e. lies on the same diagonal as) a descent of P.



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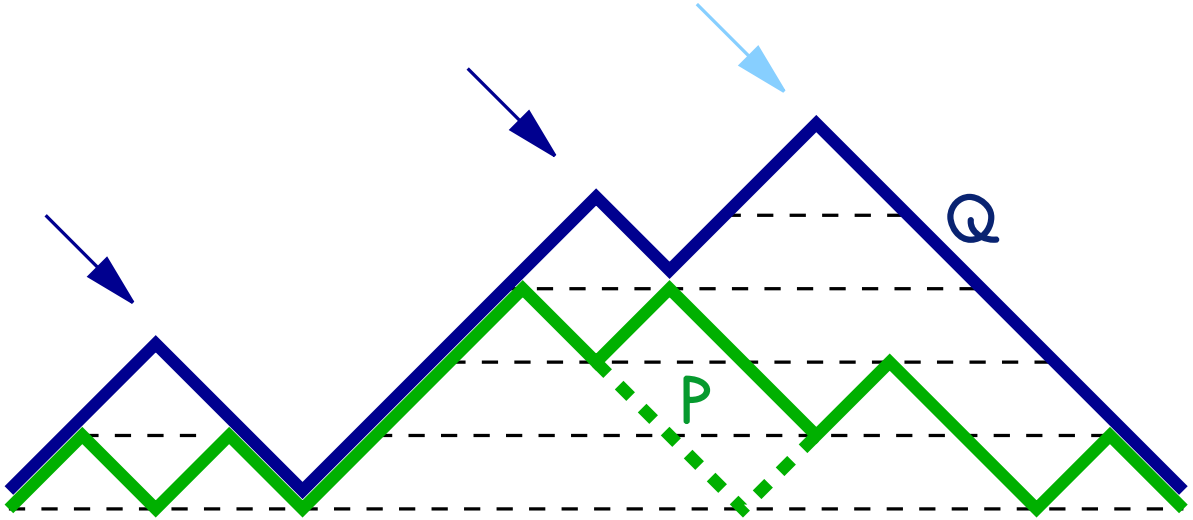

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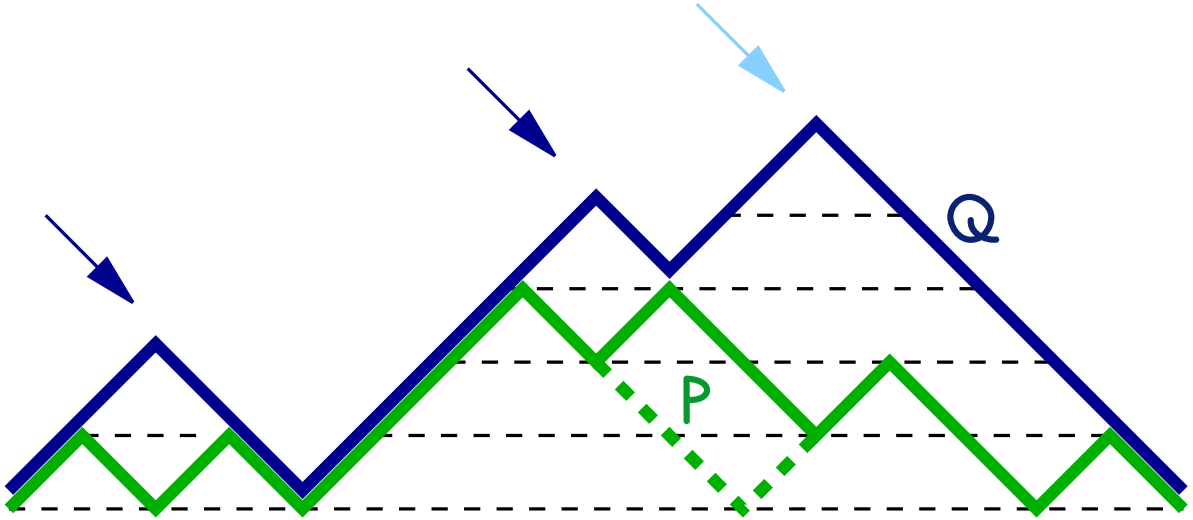


Applications:

- lattice structure
- recursive construction of intervals

II. Lattice structure

(Each pair of elements has a sup and an inf)

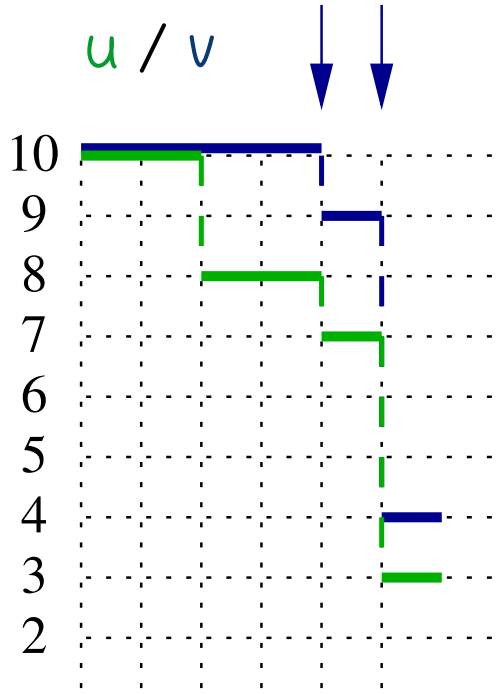
The Nadeau-Tewari poset [2024]

Def. Let $u=(u_1, \dots, u_n)$ and $v=(v_1, \dots, v_n)$ be two nonincreasing sequences of integers. Then $u \leq v$ for the NT order if

- u lies below v ($u_i \leq v_i$)
- every descent of v is a descent of u.

$v = (10, 10, 10, 10, 9, 4)$

$u = (10, 10, 8, 8, 7, 3)$



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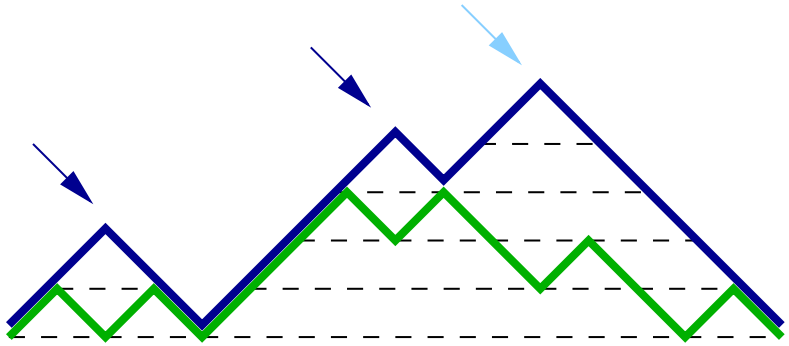

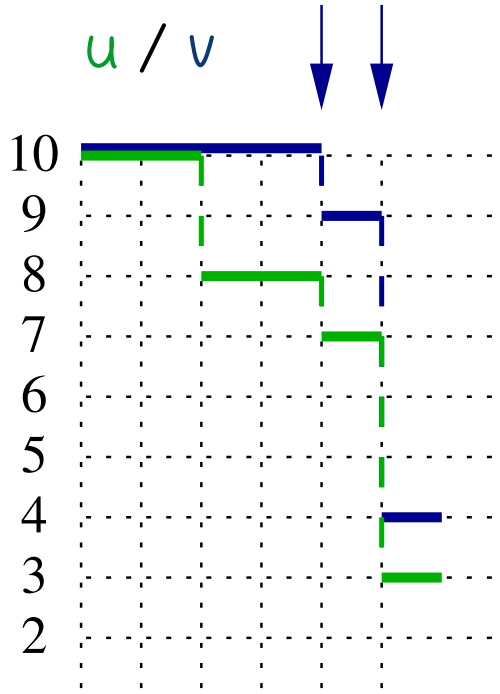

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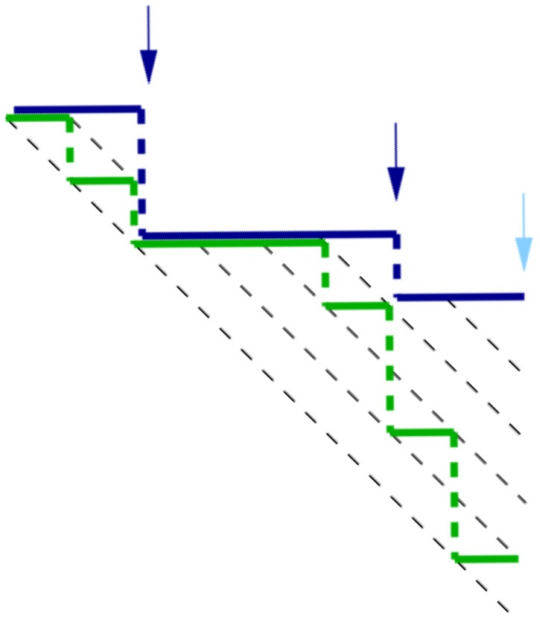

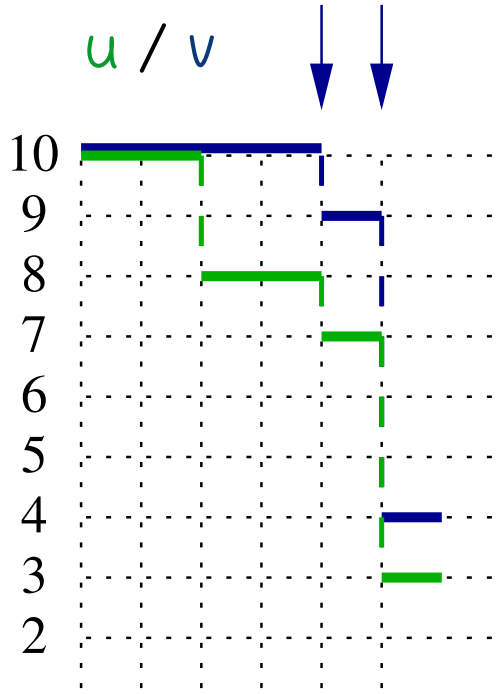

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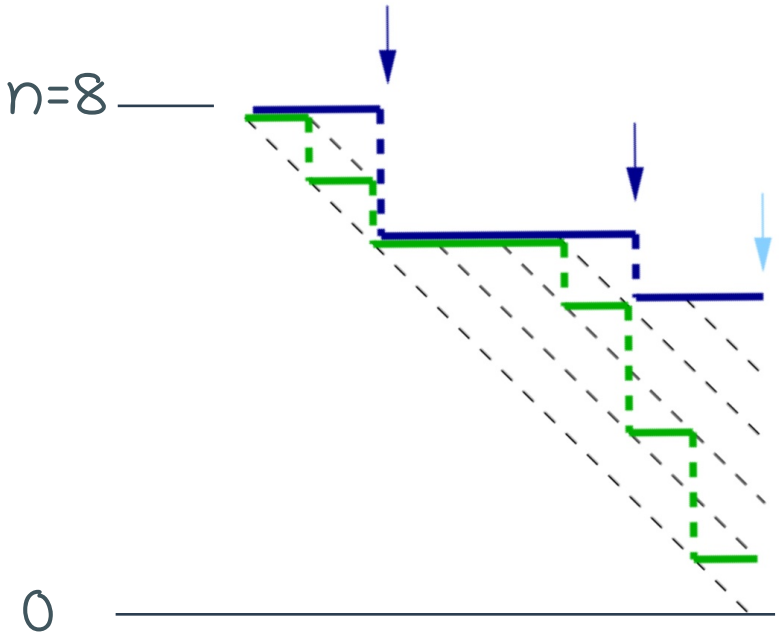
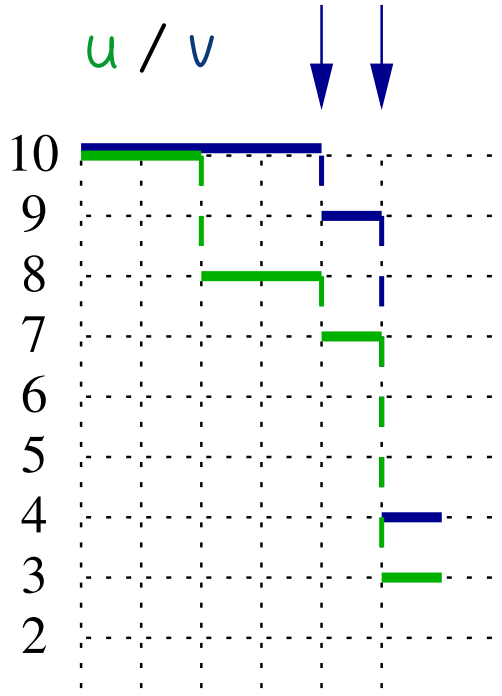


$v = (8, 8, 6, 6, 6, 6, 5, 5)$

$u = (8, 7, 6, 6, 6, 5, 3, 1)$



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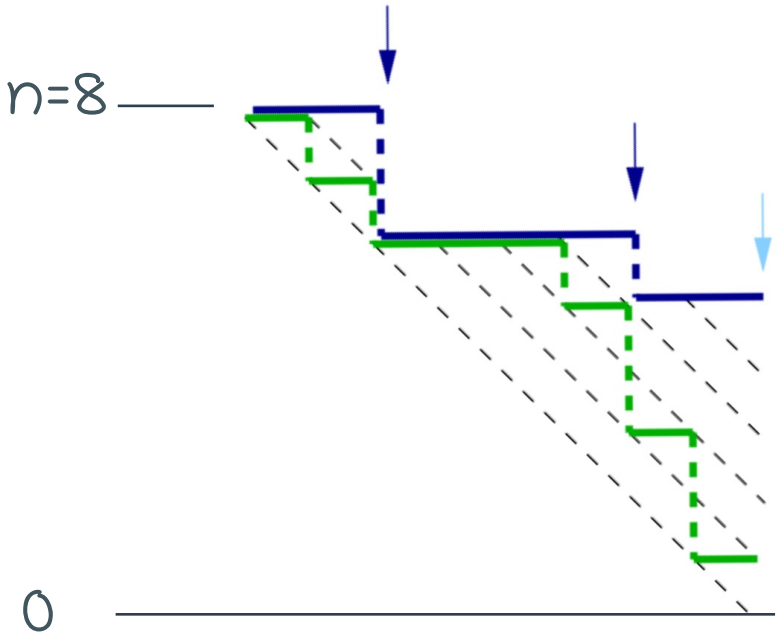
- every descent of v is a descent of u.

Observation: Dyck paths of size n can be encoded by nonincreasing sequences of length n, and then the ascent order coincides with the NT order.



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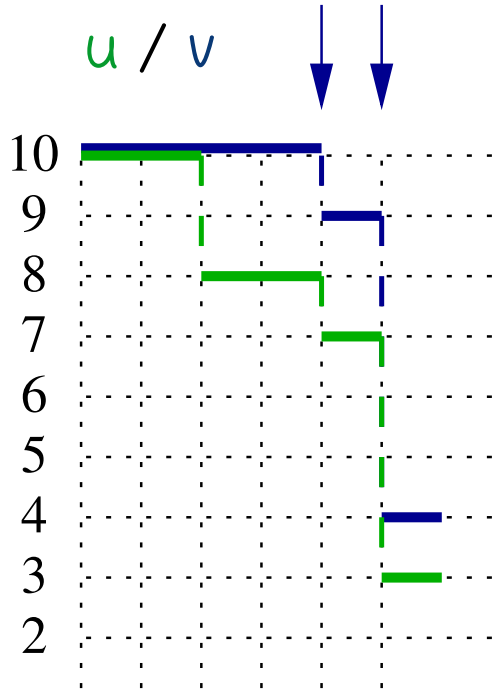
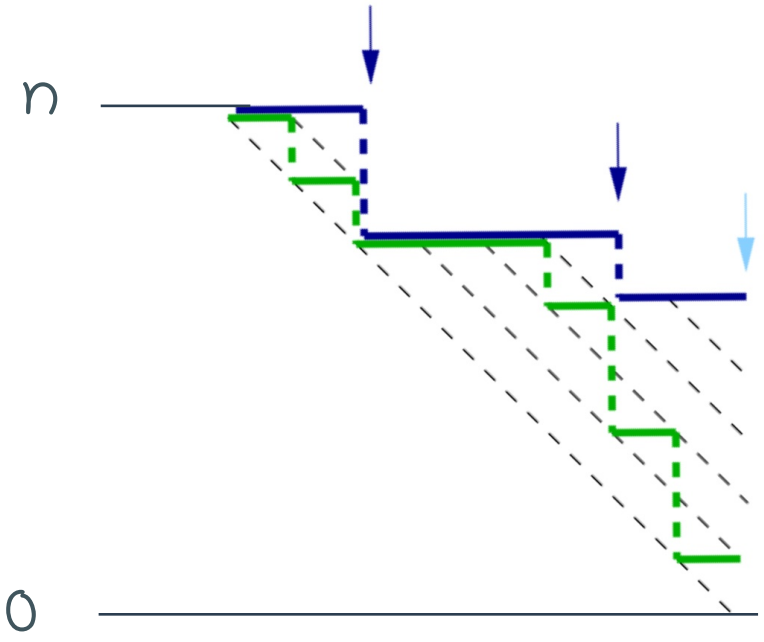


The Nadeau-Tewari poset and the ascent order

Observation: The ascent order is the order induced by the NT order on sequences $u=(u_1, \dots, u_n)$ such that $n-i+1 \leq u_i \leq n$ for all i.



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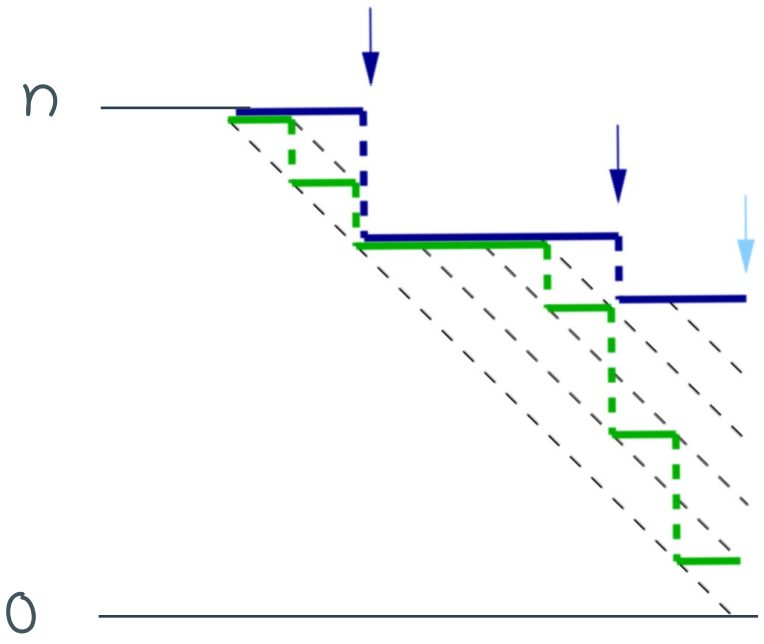
Observation: The ascent order is the order induced by the NT order on sequences $u=(u_1, \dots, u_n)$ such that $n-i+1 \leq u_i \leq n$ for all i.

These sequences form an **interval** with $u_{min} = (n, n-1, \dots, 1)$ and $u_{max} = (n, n, \dots, n)$.



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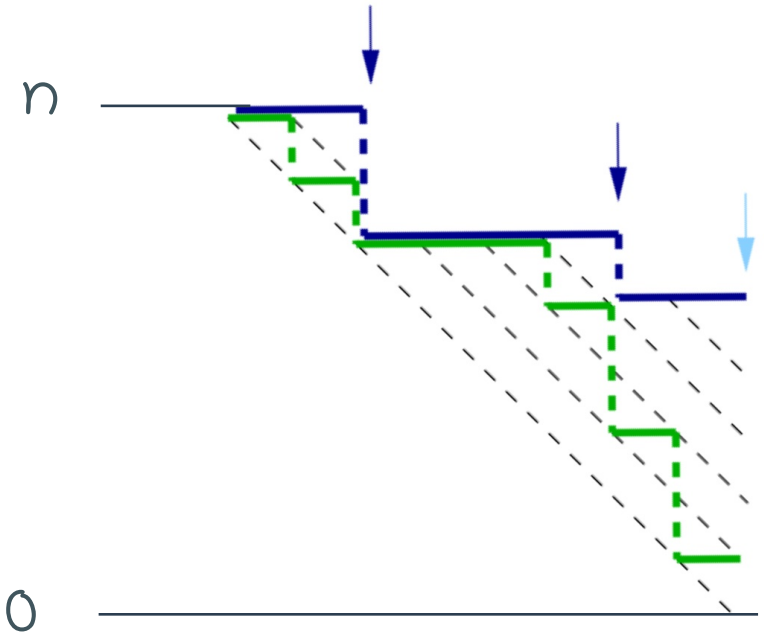
Proposition [NT 24]: the NT poset is a **lattice**.



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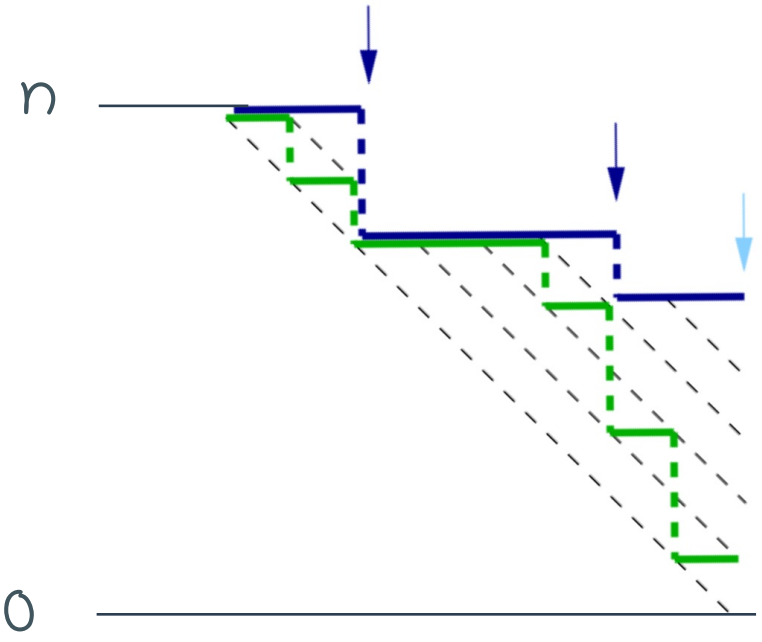
Proposition [NT 24]: the NT poset is a **lattice**.

Corollary. The ascent order on Dyck paths of size n defines a **lattice**.



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The Nadeau-Tewari poset and the ascent order

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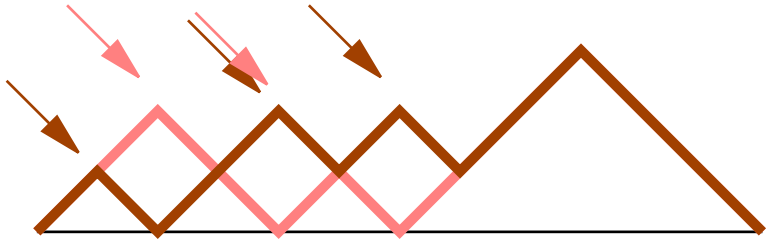
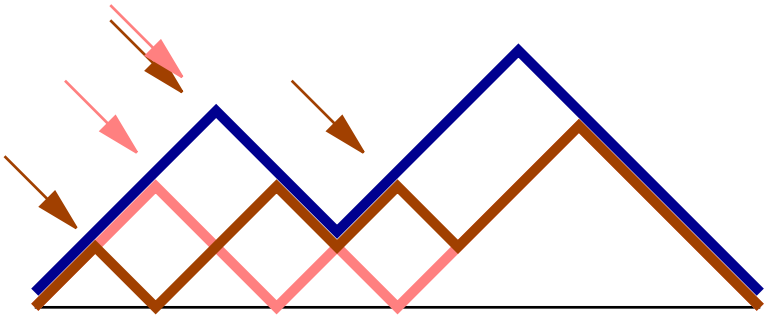
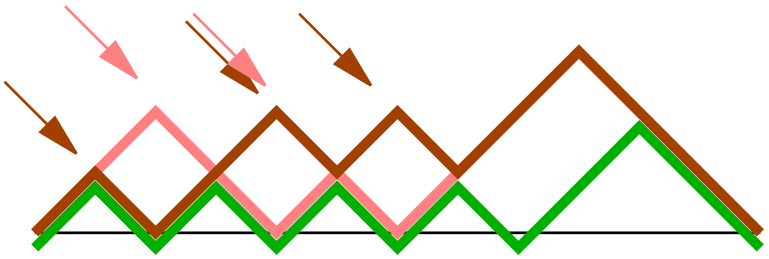


P, Q



join

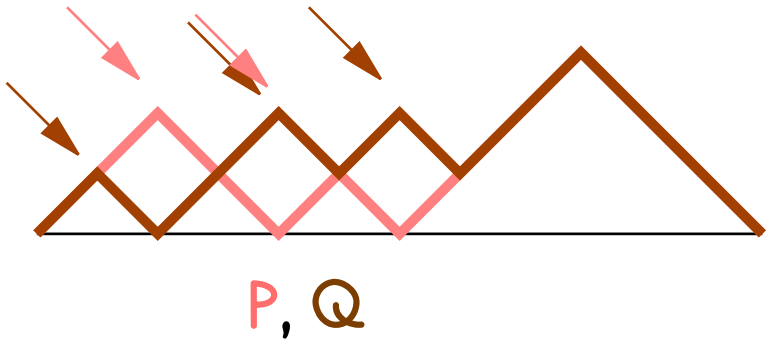


meet

The Nadeau-Tewari poset and the ascent order

Corollary. The ascent order on Dyck paths of size n defines a **lattice**.

Join(P,Q): the lowest Dyck path above P and Q whose descents are included in des(P) and in des(Q).



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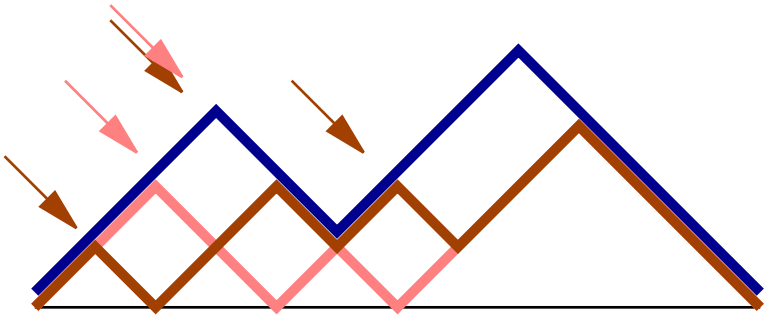

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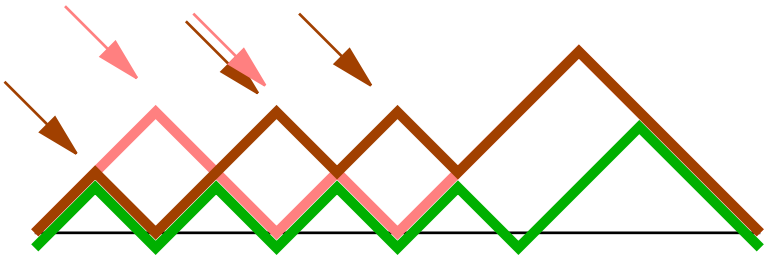

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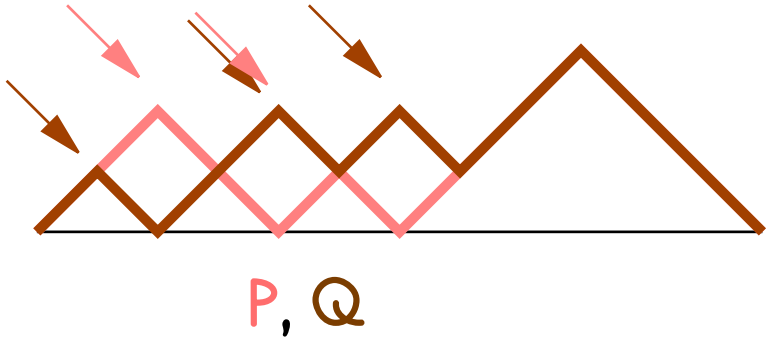

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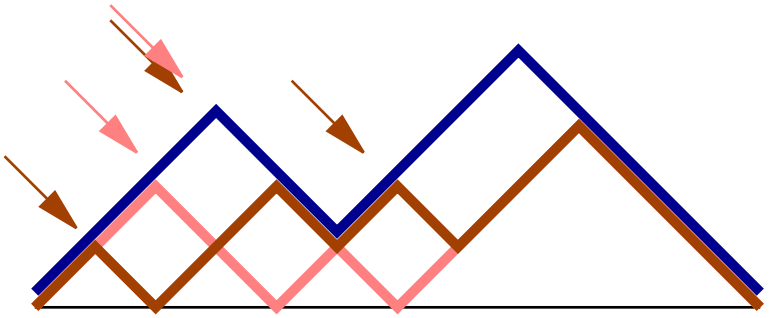

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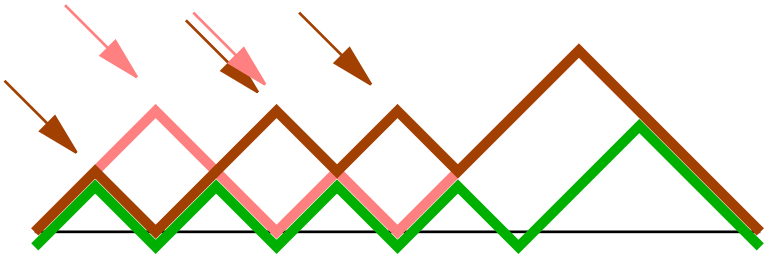


meet

III. The number of intervals

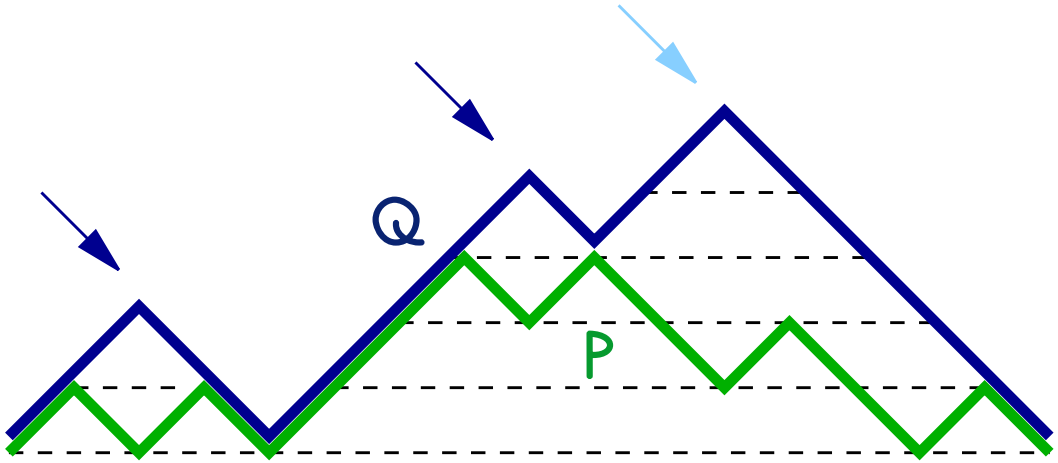
Interval $[P,Q] \sim (P,Q)$ with $P \preccurlyeq Q$

Recursive construction of ascent intervals

Recursive construction of ascent intervals

Proposition. In the ascent poset, $P \preccurlyeq Q$ iff

- P lies below Q
- every descent of Q is (i.e. lies on the same diagonal as) a descent of P.



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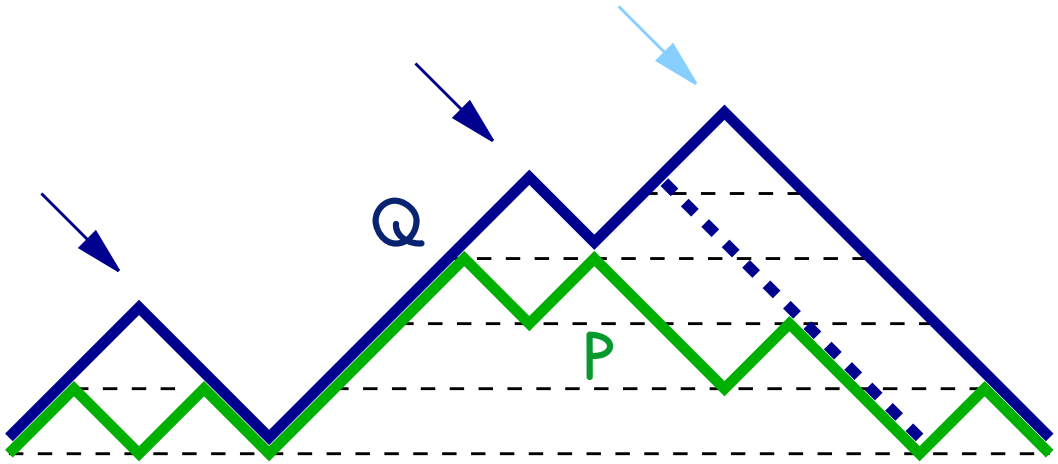

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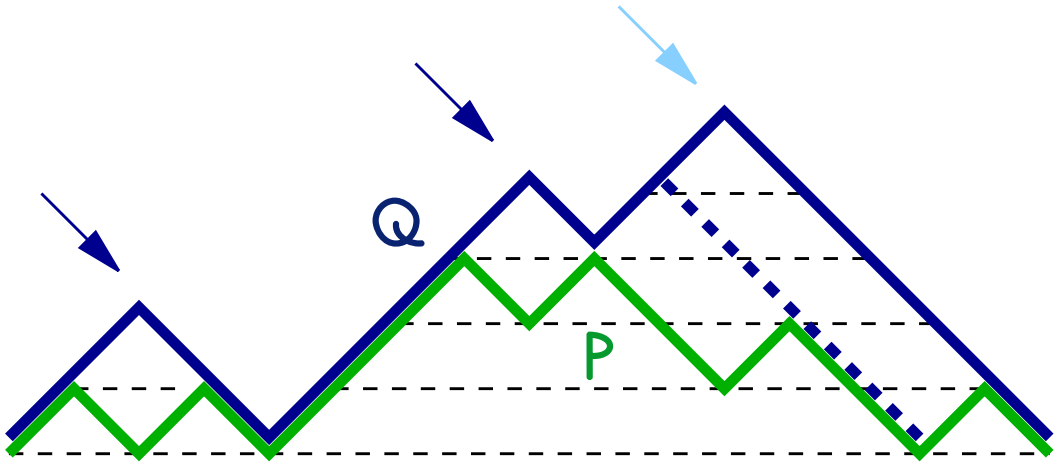

Corollary: if [P,Q] is an interval, deleting the last peak of P and the last peak of Q gives a new interval.

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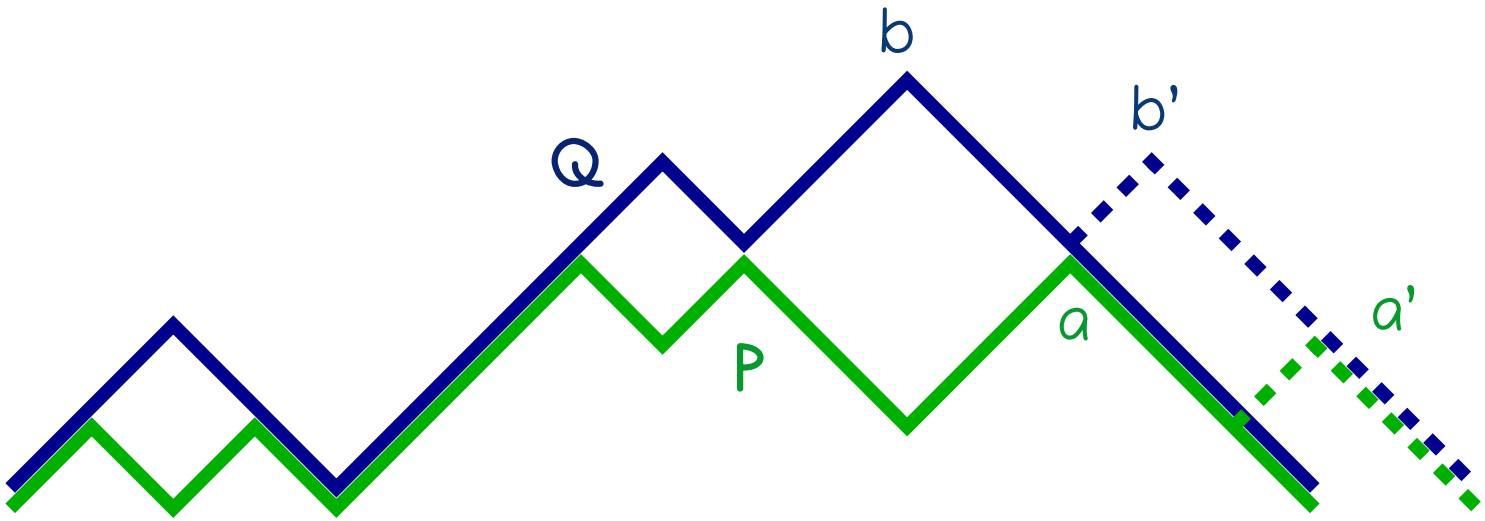
Conversely, starting from an interval [P,Q] with final peaks at heights $a \leq b$, adding peaks in P and Q at heights a' and b' gives an interval iff...



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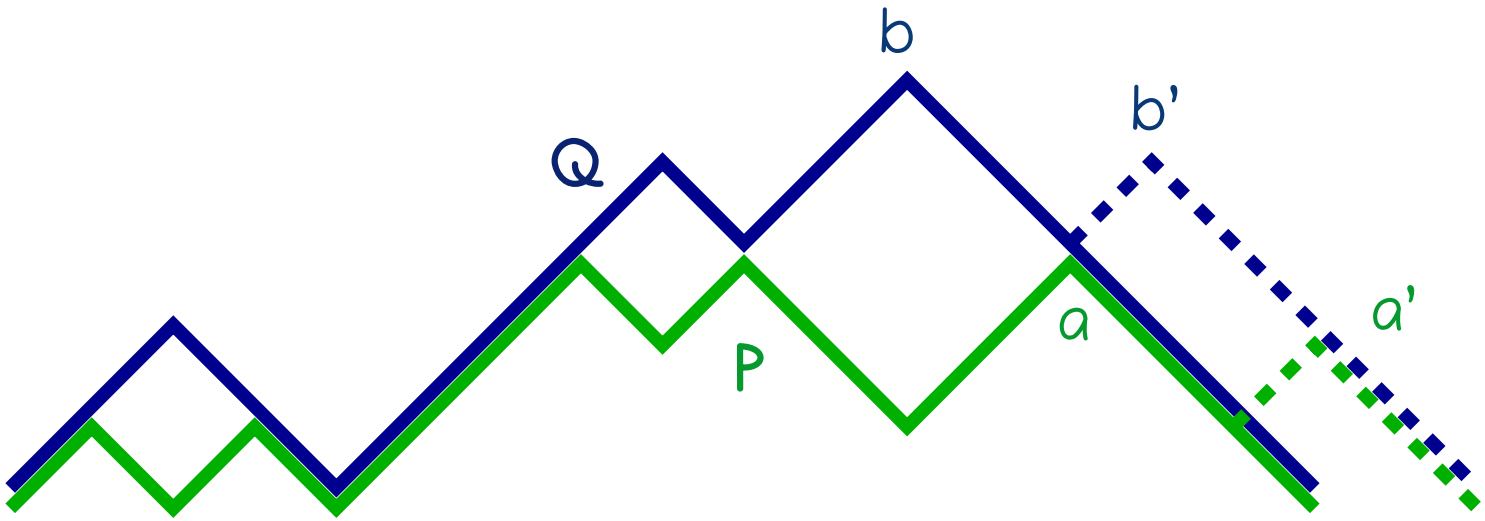

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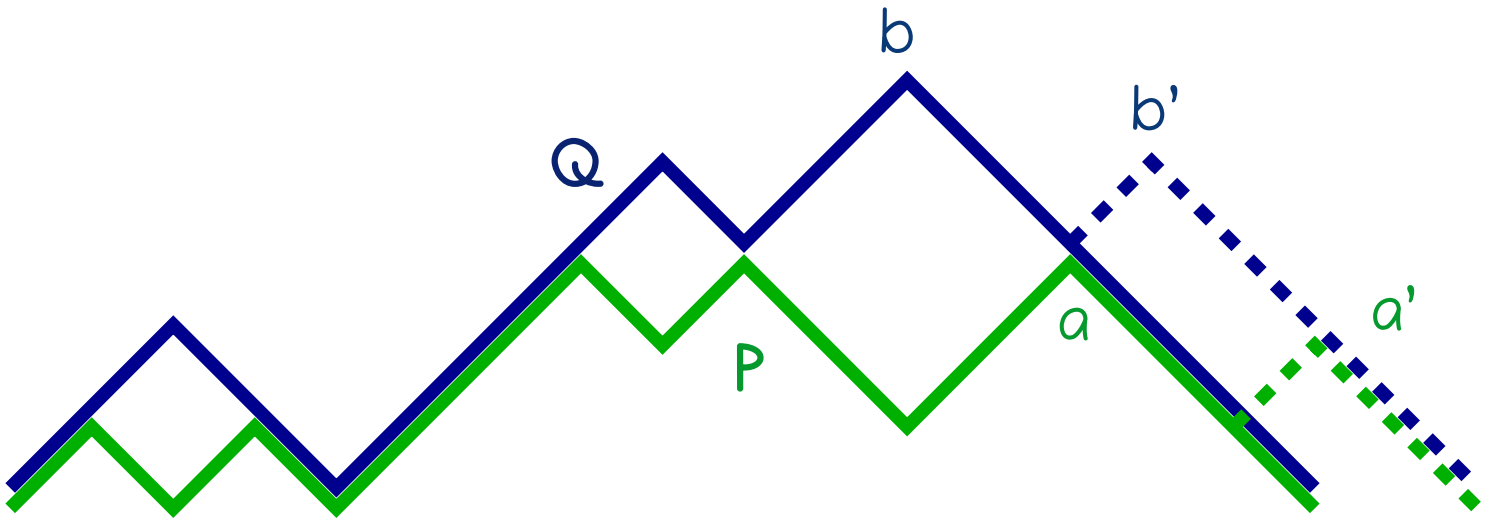

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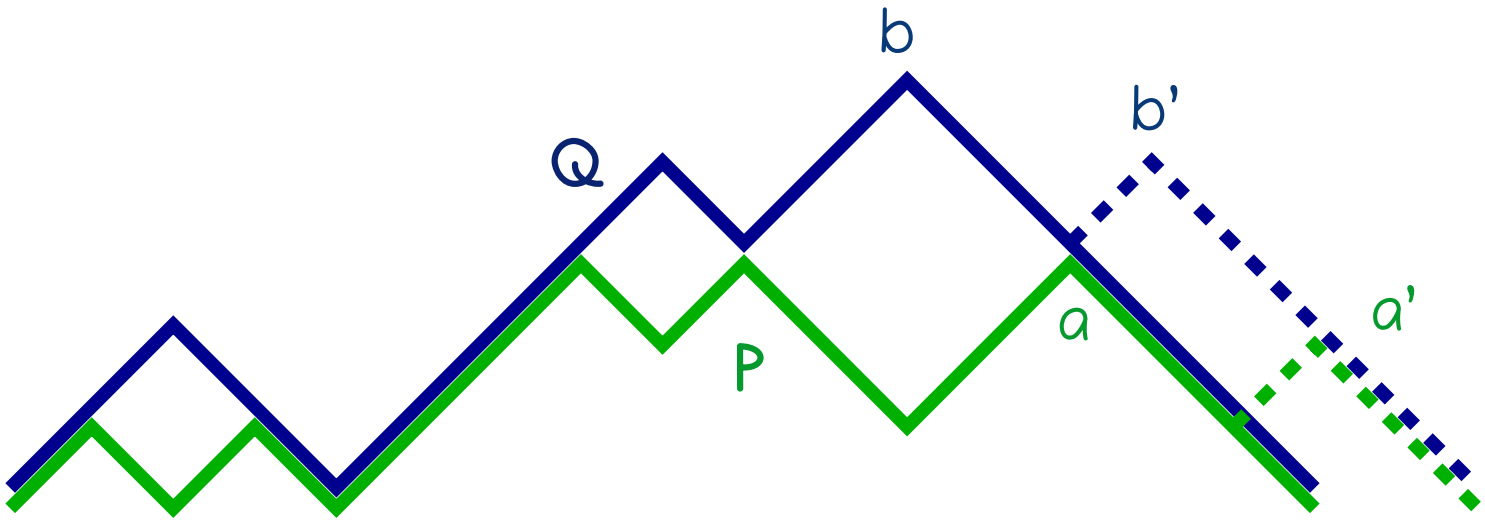

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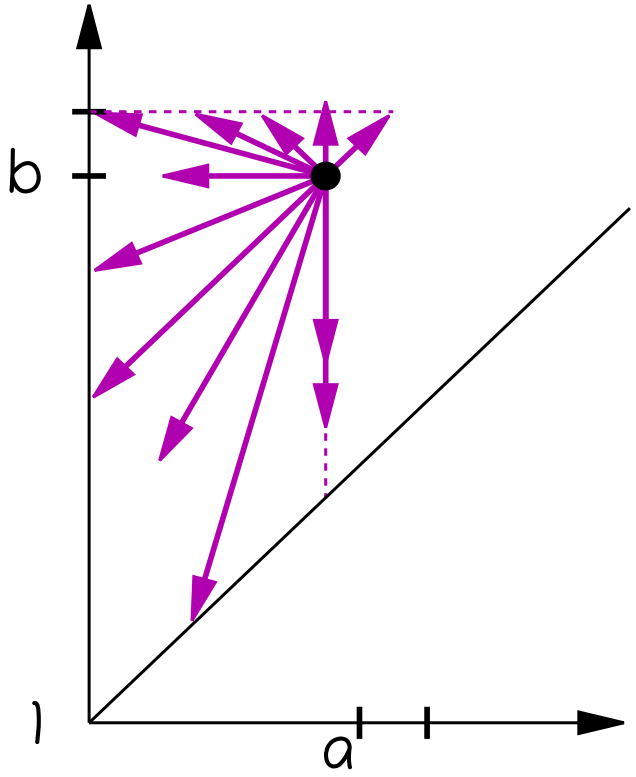

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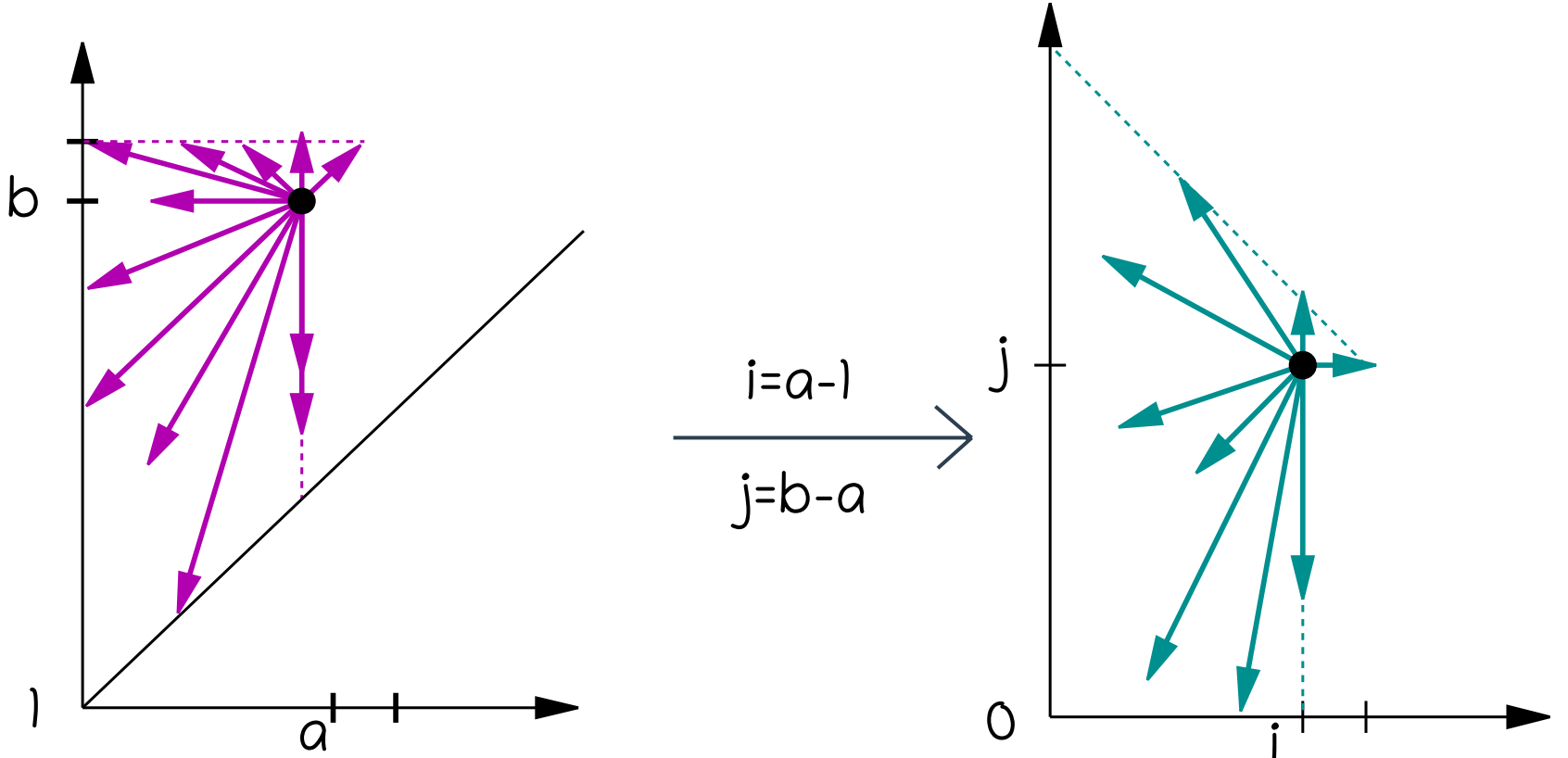

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intervals of size n

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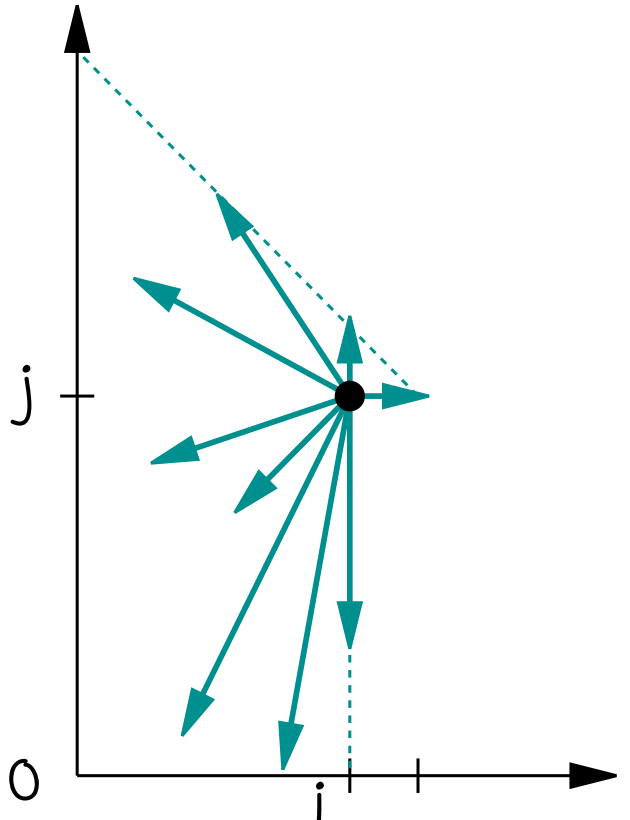
quadrant walks of length n-1 starting from (0,0)

≈

quadrant walks of length n starting and ending at (0,0)



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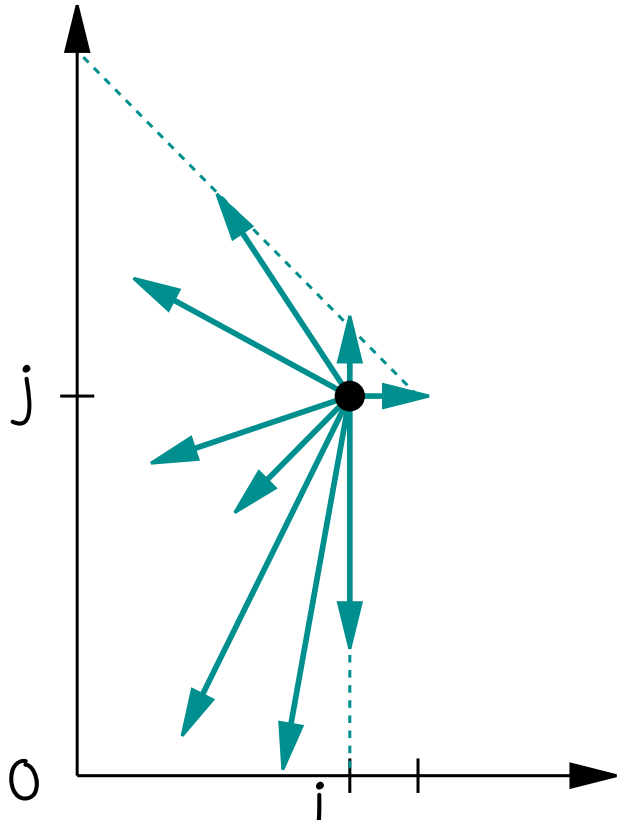

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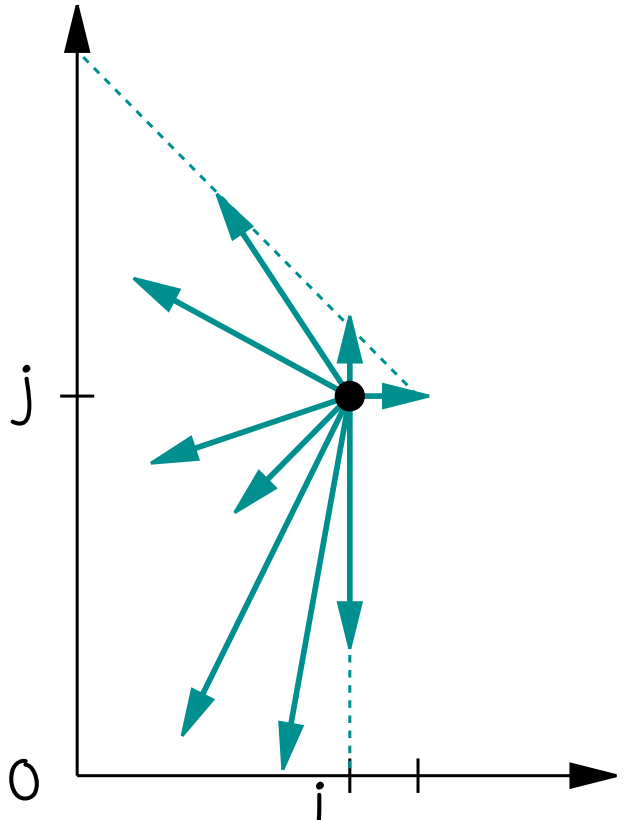

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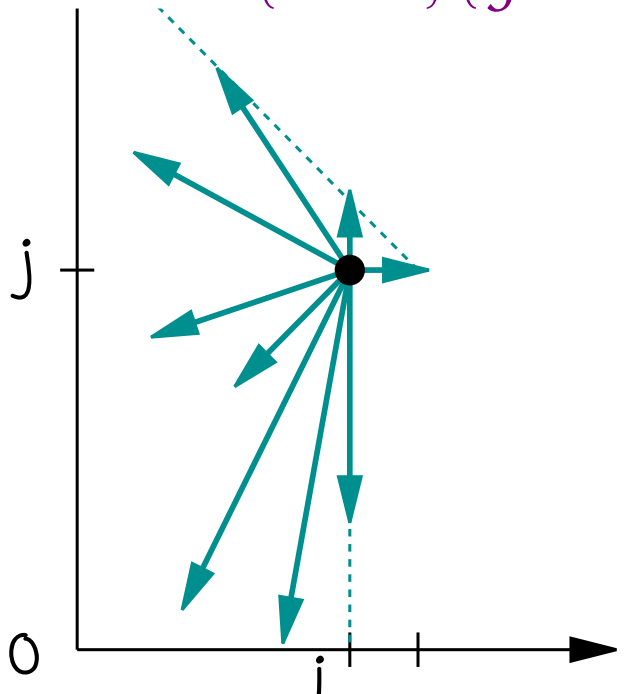

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Asymptotics:

$$g(n) \sim \kappa\,\mu^n n^{-7/2}, \qquad \text{with} \qquad \mu = \frac{11+5\sqrt{5}}{2}.$$

IV. m-Dyck paths, and mirrored m-Dyck paths

m-Dyck paths and mirrored m-Dyck paths

In an m-Dyck path, the length of all ascents is a multiple of m.

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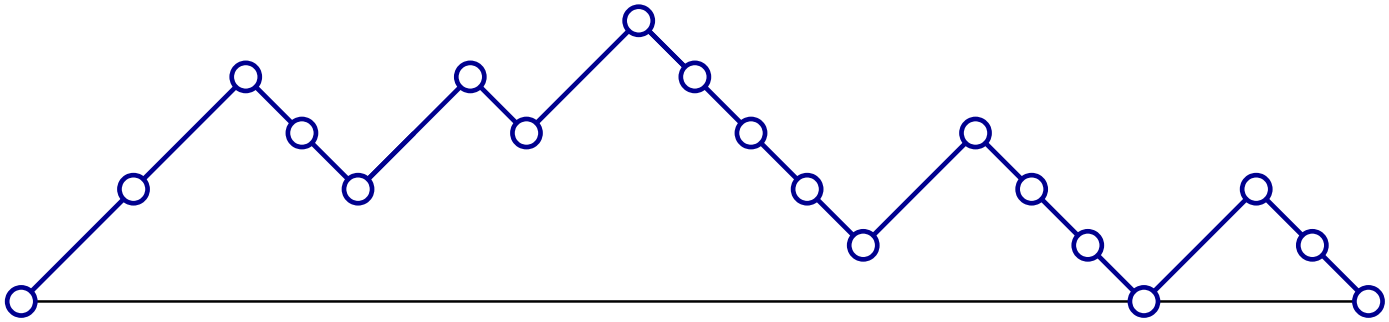

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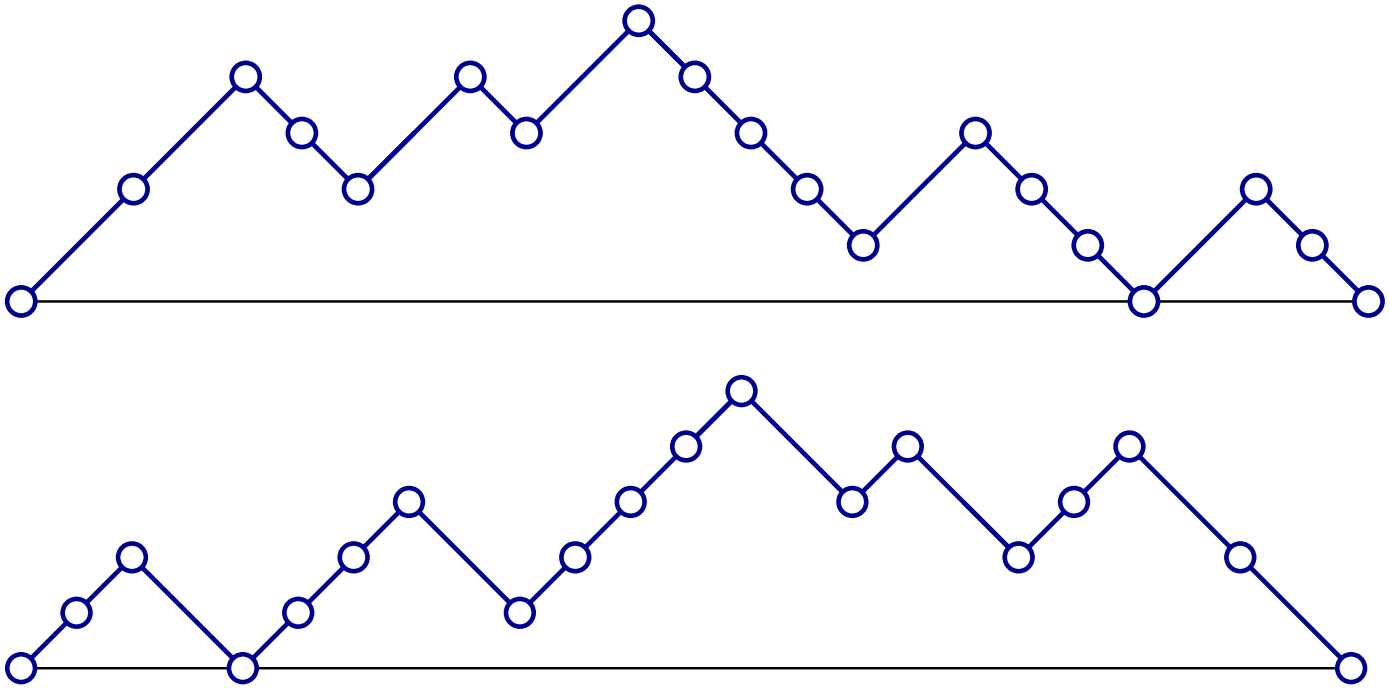

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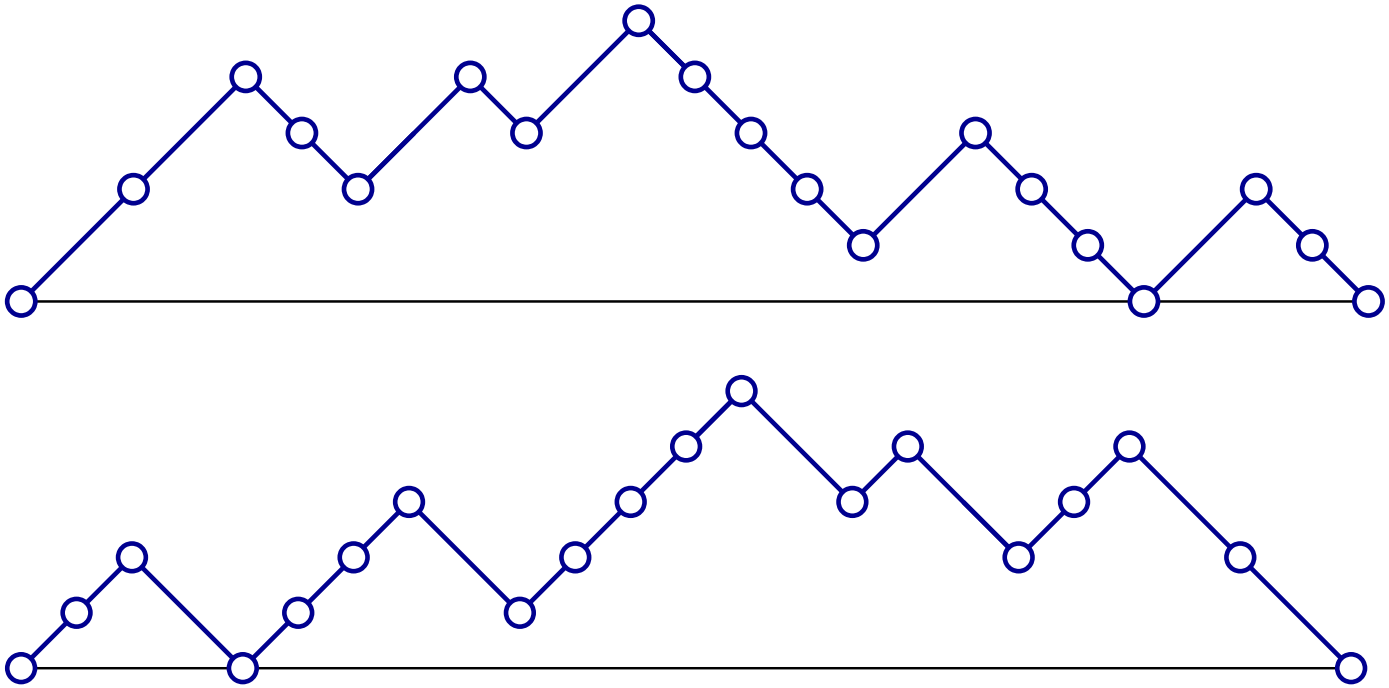
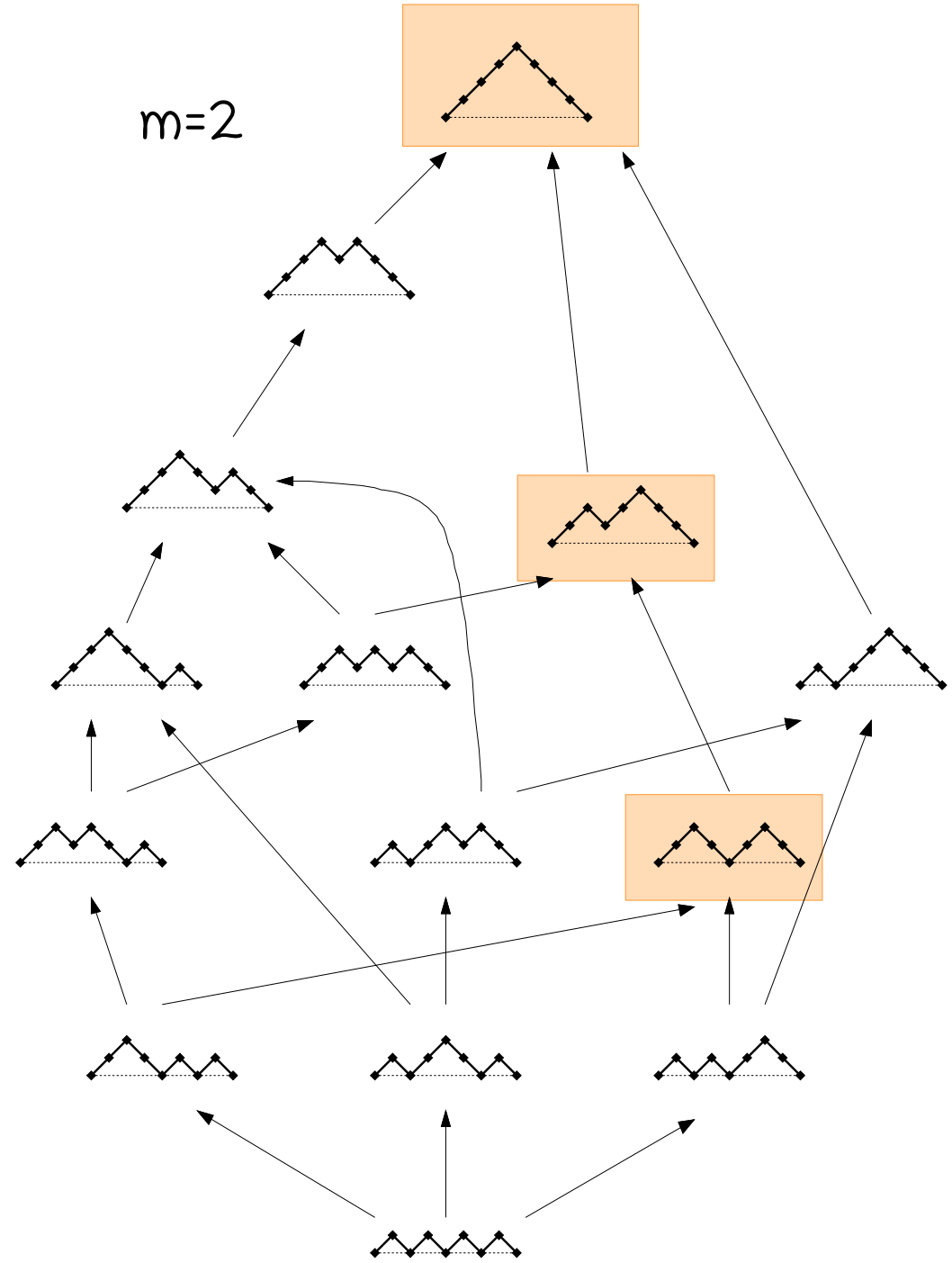


→ Study the order induced by the ascent order on m-Dyck paths and mirrored m-Dyck paths.

m-Dyck paths



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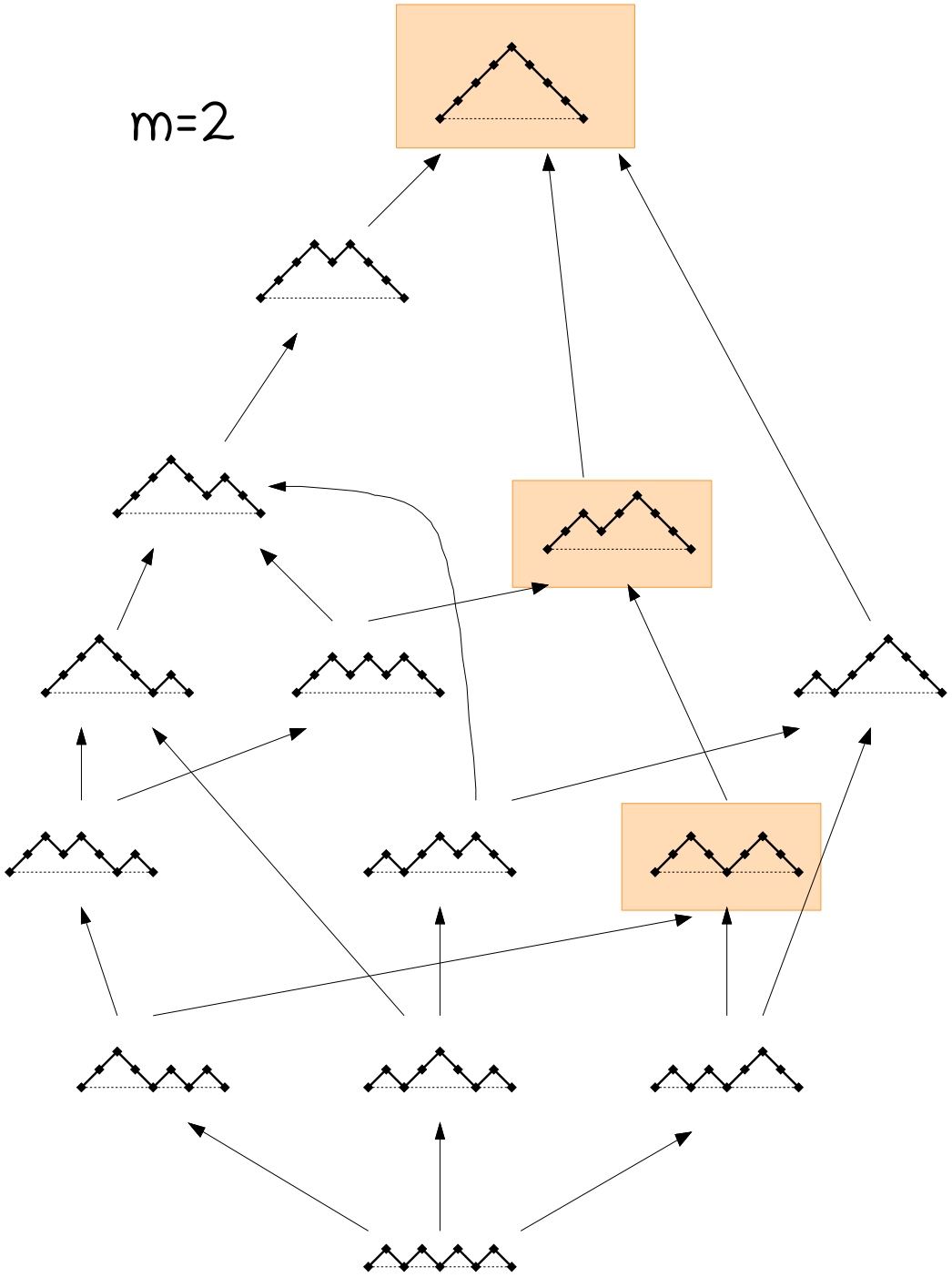


m-Dyck paths form an interval in the ascent lattice A_{mn}, with

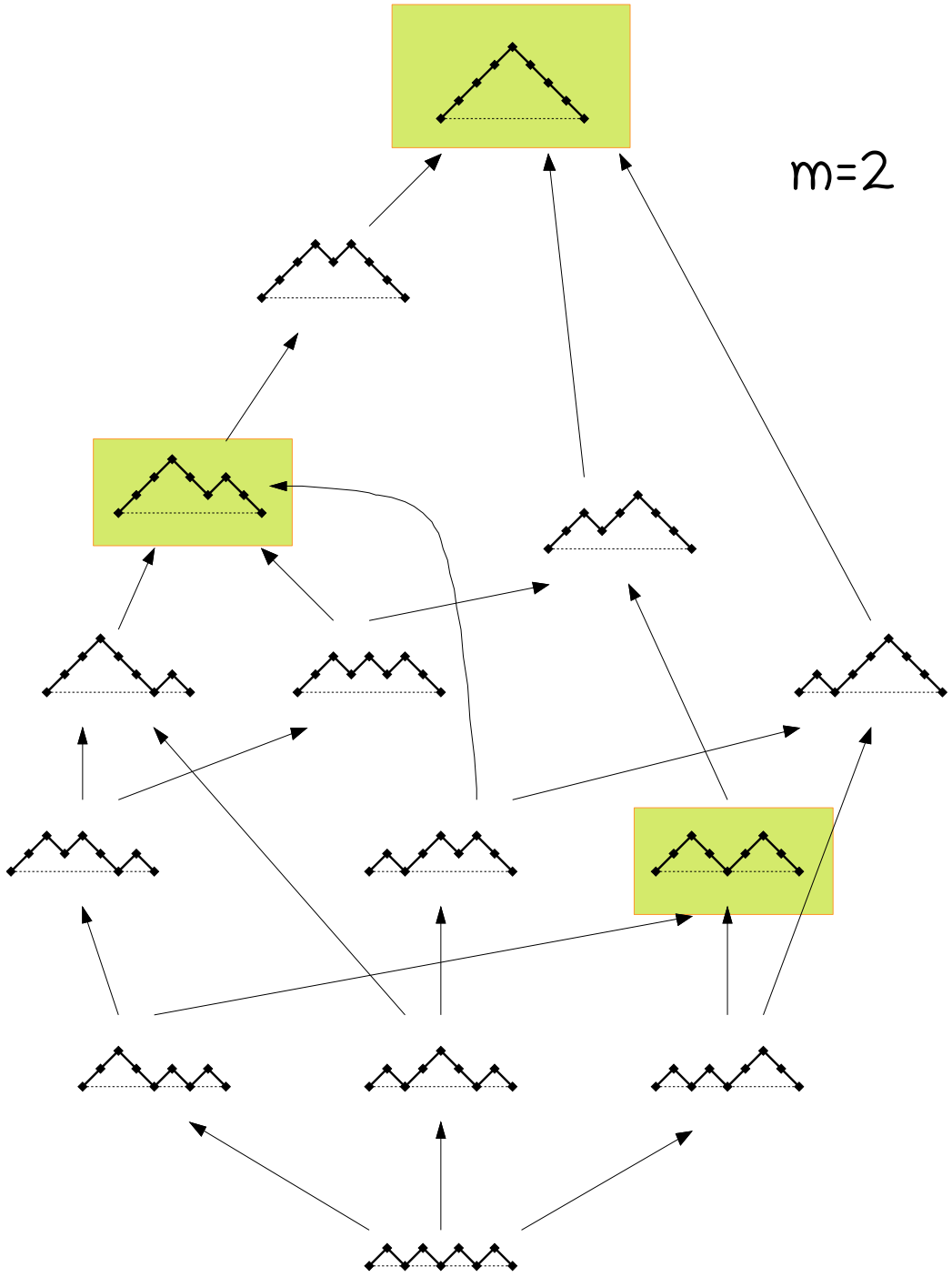
$$\min = (U^m D^m)^n$$

$$\max = U^{mn} D^{mn}.$$

In particular, it is a lattice.

Mirrored m-Dyck paths

Mirrored m-Dyck paths only form a join semi-lattice.



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Intervals in m-Dyck paths:

- Stanley lattice: D-finite GF (i.e., linear DE with pol. coeffs)

$$\frac{2(m+2)((m+1)n)!((m+1)(n+1))!}{n!(n+1)!(mn+2)!(m(n+2)+2)!}$$

- Tamari lattice: algebraic GF [mbm, Fusy, Préville-Ratelle 11]

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}$$

Conj: Bergeron, Préville-Ratelle

- Greedy Tamari lattice: algebraic GF [mbm, Chapoton 24]

$$\frac{(m+2)(m+1)^{n-1}}{(mn+1)(mn+2)}\binom{(m+1)n}{n}.$$

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Intervals in mirrored m-Dyck paths:

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For m>1, the exponent α is irrational, and hence the GF of intervals cannot be D-finite.

[Bostan, Raschel, Salvy 14]

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V. Connection with the sylvester congruence

[Hivert, Novelli, Thibon 05]

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Search: **seq:1,3,13,69,417,2759 id:243688**

Displaying 1-1 of 1 result found.

Sort: relevance | references | number | modified | created Format: long | short | data

A243688 Number of Sylvester classes of 1-multiparking functions of length n.

1, 3, 13, 69, 417, 2759

(list; graph; refs; listen; history; text; internal format)

OFFSET 1,2

COMMENTS See Novelli-Thibon (2014) for precise definition.

LINKS Table of n, a(n) for n=1..6.

J.-C. Novelli, J.-Y. Thibon, Hopf Algebras of m-permutations, (m+1)-ary trees, and m-parking functions, arXiv preprint arXiv:1403.5962, 2014. See Fig. 26.

KEYWORD nonn,more

AUTHOR N. J. A. Sloane, Jun 14 2014

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Search: **seq:1,5,40,407,4797 id:243671**

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Link with the NT-lattice

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STATUS approved

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- Defined on words on the alphabet $\mathbb{Z}$
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4 2 5 3 3 → 4 5 2 3 3, sylvester word

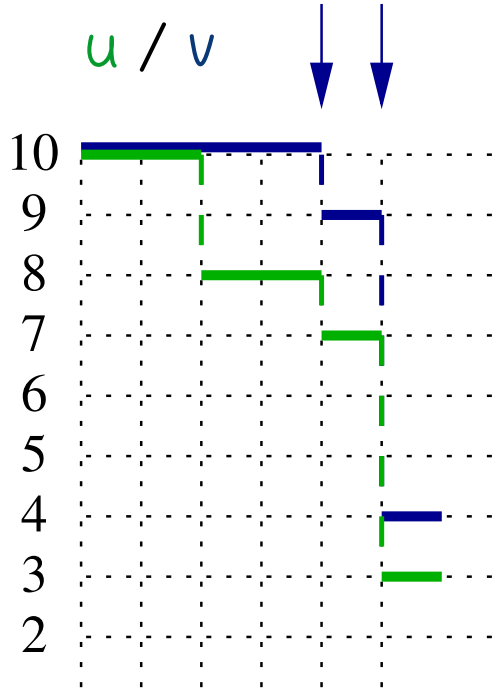
The Nadeau-Tewari lattice [2024]

Def. Let $u=(u_1, \dots, u_n)$ and $v=(v_1, \dots, v_n)$ be two nonincreasing sequences of integers. Then $u \leq v$ for the NT order if

- u lies below v ($u_i \leq v_i$)
- every descent of v is a descent of u.

$v = (10, 10, 10, 10, 9, 4)$

$u = (10, 10, 8, 8, 7, 3)$



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Example. Fix n=6 and a sylvester word w on the alphabet {1, 2, ..., n}, containing the letter 1, say w = **3 2** 2 2 2 5 **1** 1 1 5.

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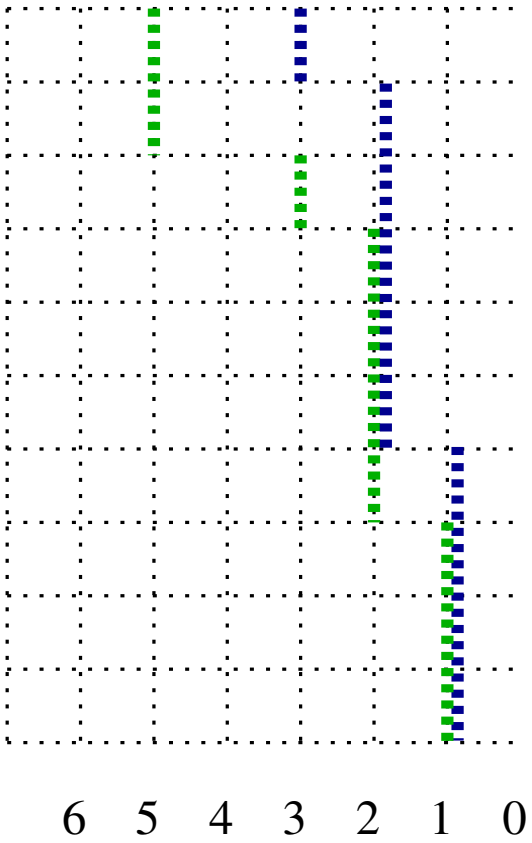

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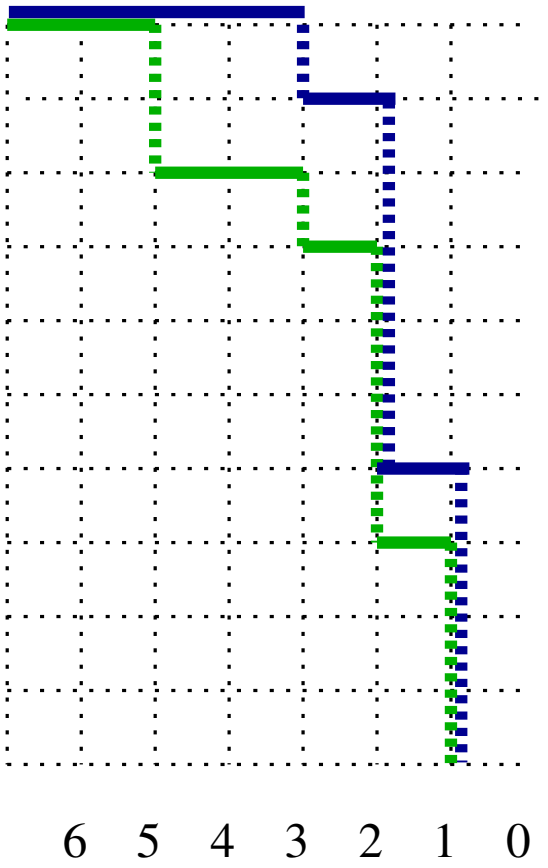

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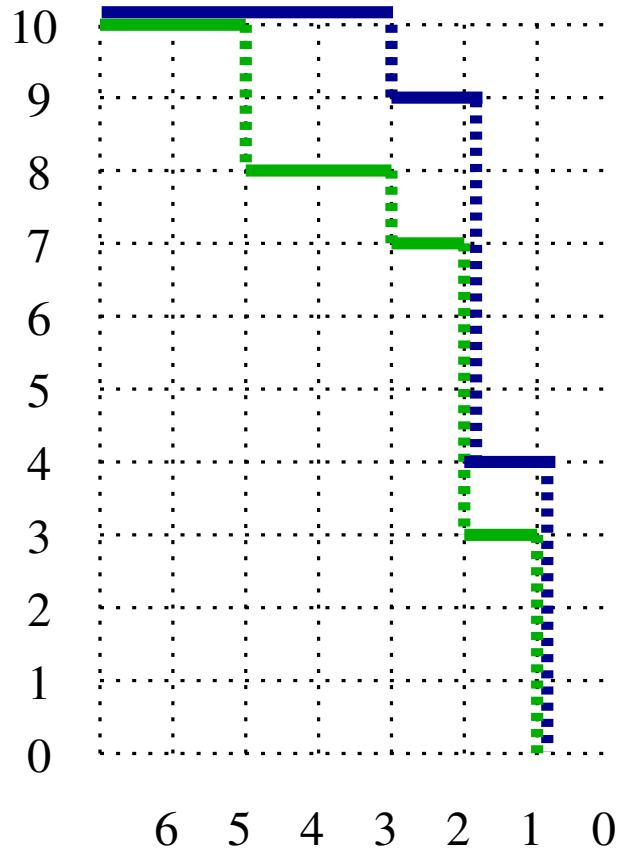
- The horizontal words u = 10 10 8 8 7 3 and v = 10 10 10 10 9 4, of length n=6, form an interval in the Nadeau-Tewari lattice.



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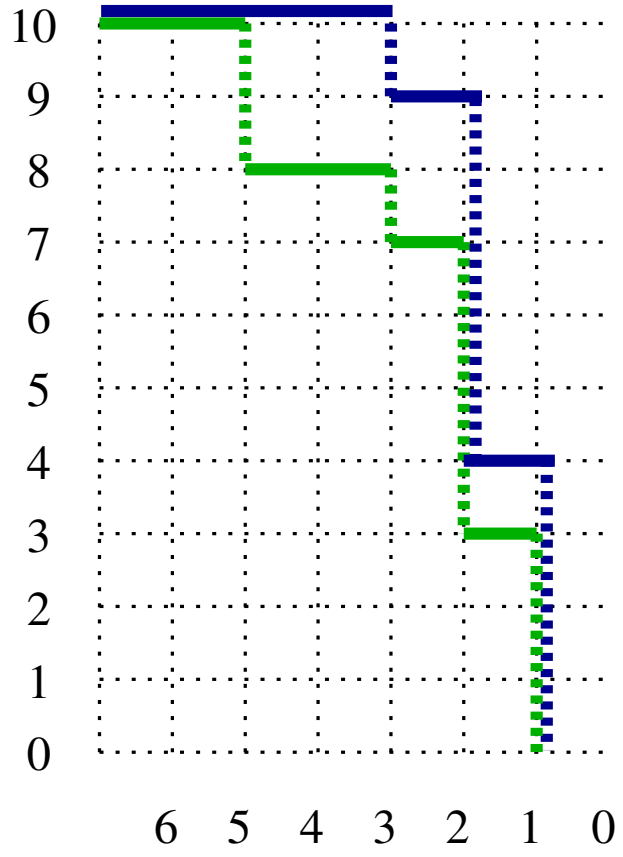

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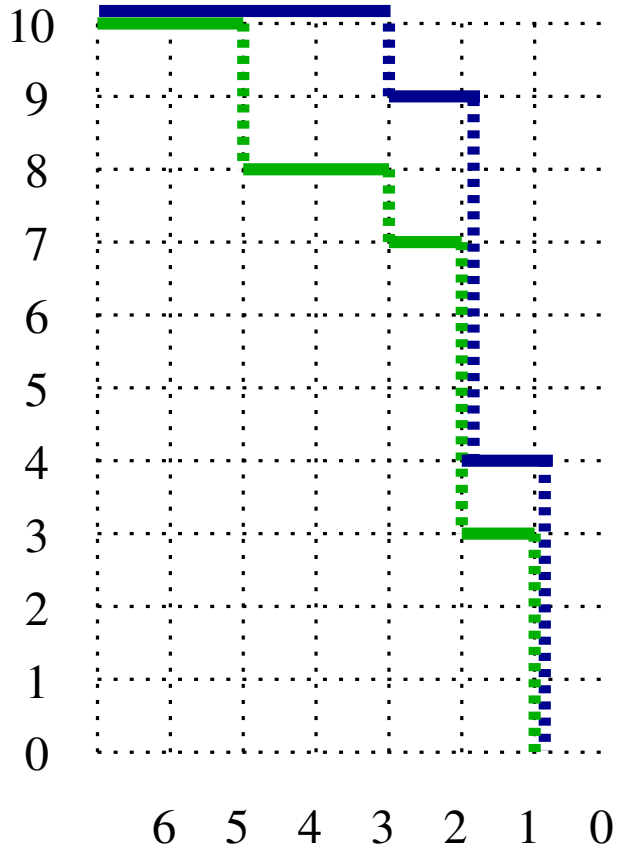

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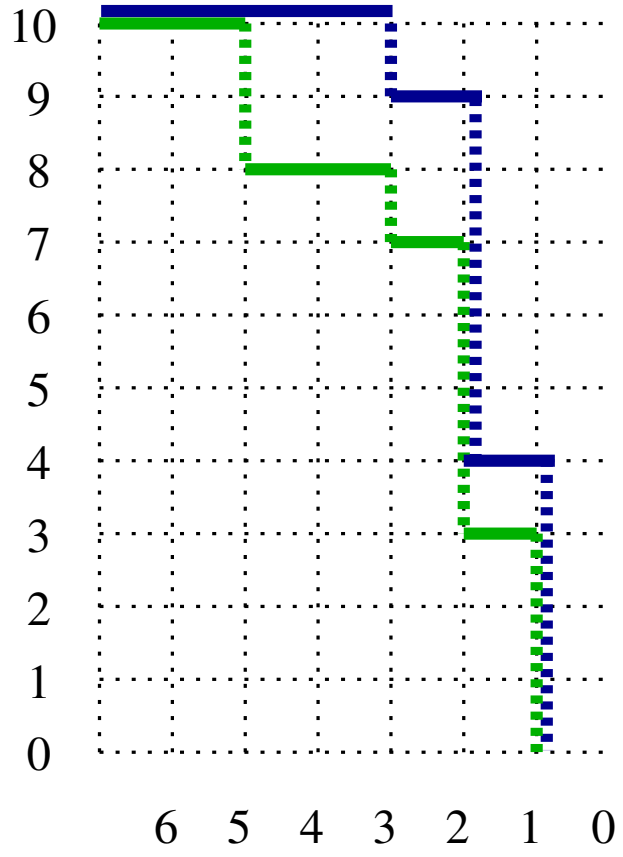

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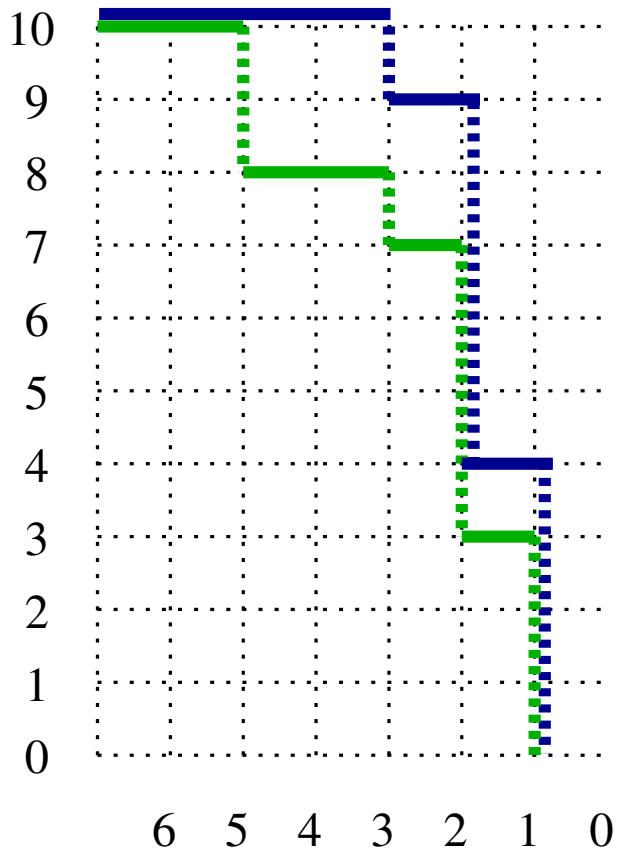

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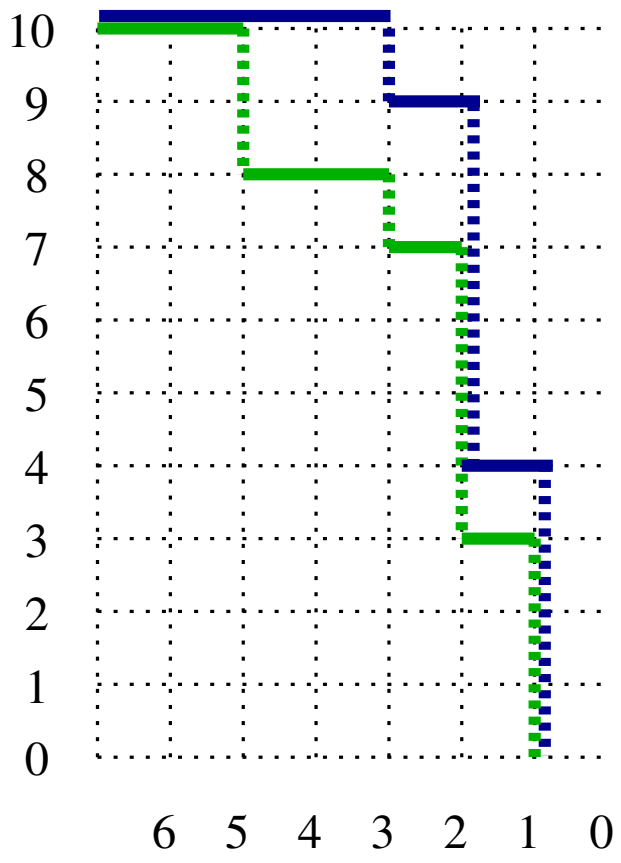

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Specializations: bijections between

- positive sylvester words w of length mn such that $\mathrm{Inv}(w) \leq n^m (n-1)^m \dots 2^m 1^m$ and ascent intervals of m-Dyck paths of length mn
- positive sylvester words w of length n such that $\mathrm{Inv}(w) \leq ((n-1)m+1) \dots (2m+1)(m+1)1$ and ascent intervals of mirrored m-Dyck paths of length mn.

VI. Final remarks

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- **Combinatorial proof** for the number/GF of ascent intervals? (m=1)

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- **A symmetric joint distribution** on ascent intervals [P,Q] (m=1):

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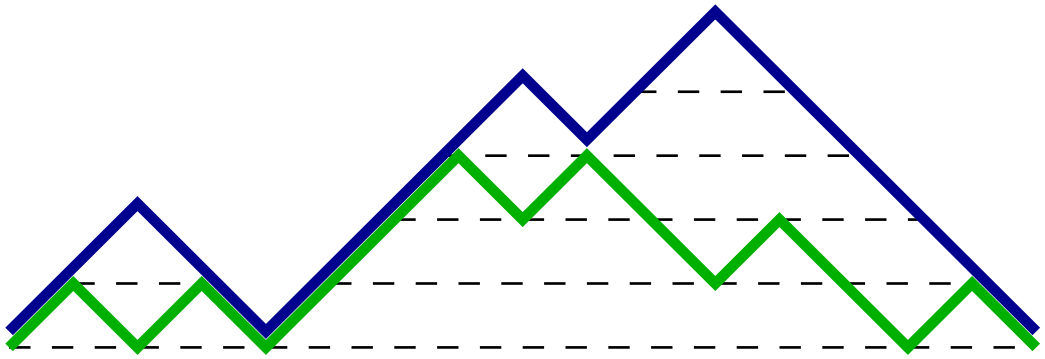
Example. a(P)=1, r(P,Q)=2

(Non-recursive) **bijection**?



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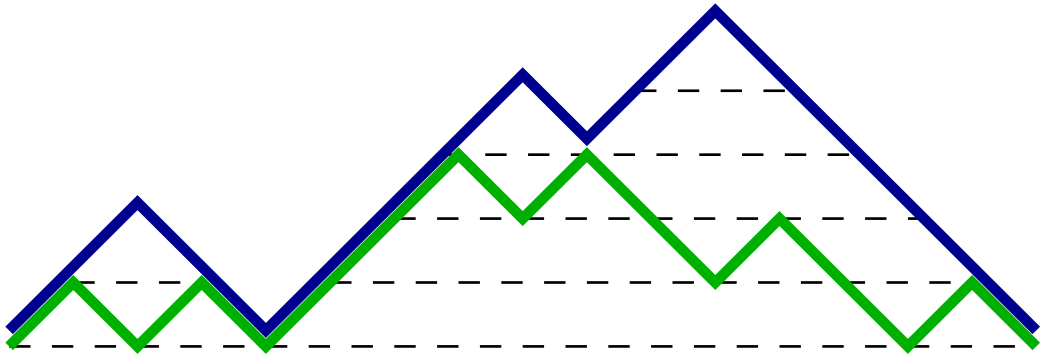

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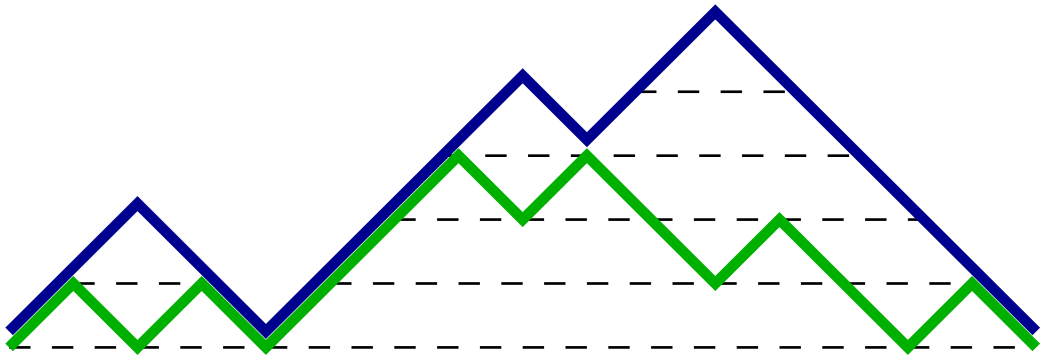

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Thanks for your attention

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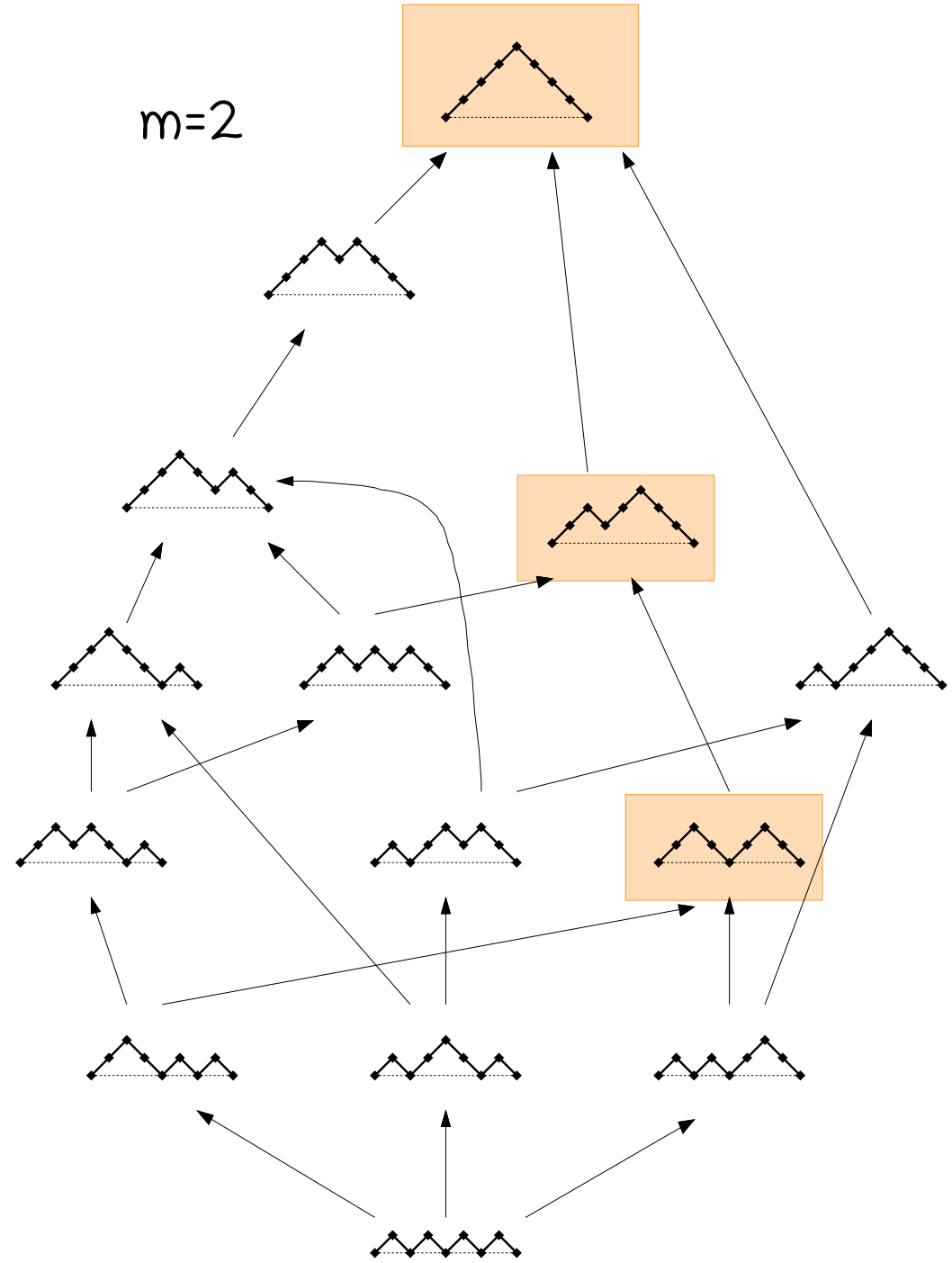
m-Dyck paths and mirrored m-Dyck paths



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