

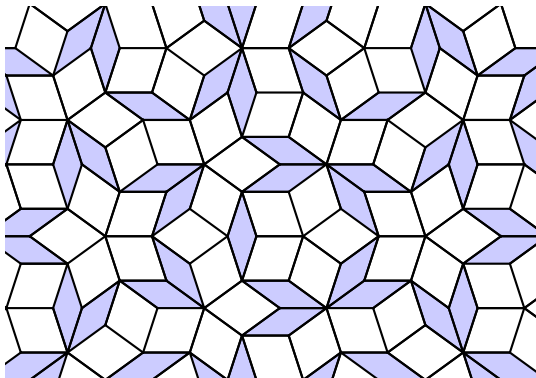
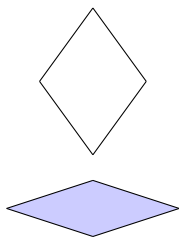
# Aperiodic tilesets derived from Cut and project tilings

Carole Porrier

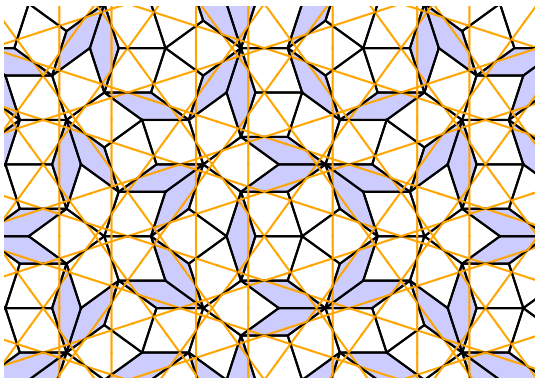
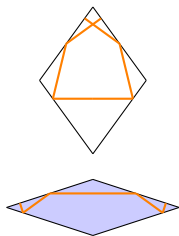
Université Sorbonne Paris Nord  
Université du Québec à Montréal

Séminaire CODYS, Bordeaux  
5 décembre 2022

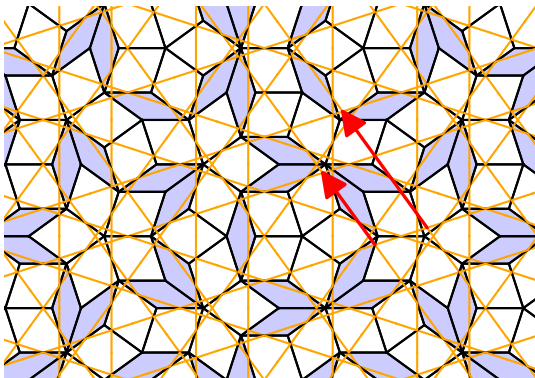
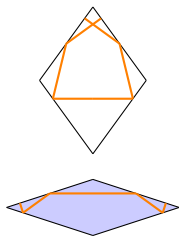
# Ammann bars in Penrose tilings



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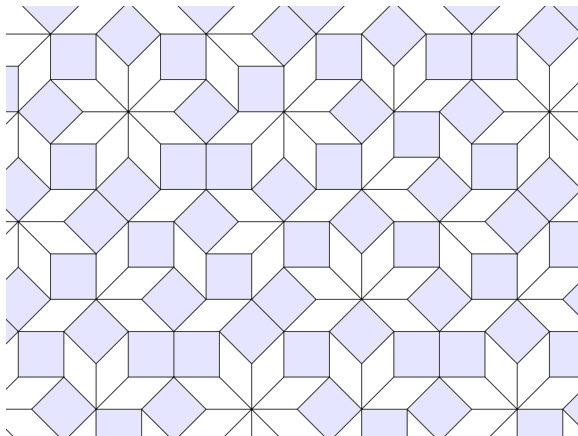
# Ammann bars in Penrose tilings



# Tiling with parallelograms

$n$  pairwise non-collinear vectors  $v_0, \dots, v_{n-1}$  ( $n \geq 3$ ) of the plane define  $\binom{n}{2}$  **prototiles**  $T_{ij} := \{\lambda v_i + \mu v_j \mid 0 \leq \lambda, \mu \leq 1\}$

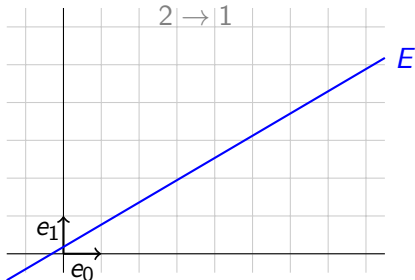
**Tiles:** translated prototiles, placed edge to edge



Cut and project  $n \rightarrow d$ 

$e_0, \dots, e_{n-1}$  canonical basis of  $\mathbb{R}^n$

$E \subset \mathbb{R}^n$  affine  $d$ -plane, called **slope**

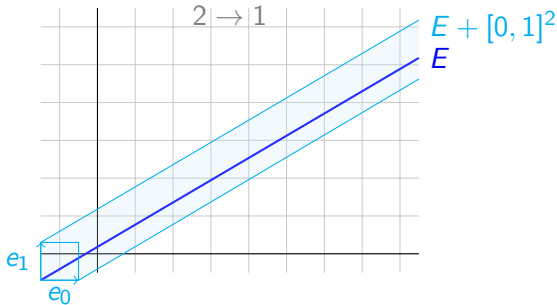


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**Cut**  $C : \{ d\text{-dim facets of } \mathbb{Z}^n, \subset E + [0, 1]^n \}$

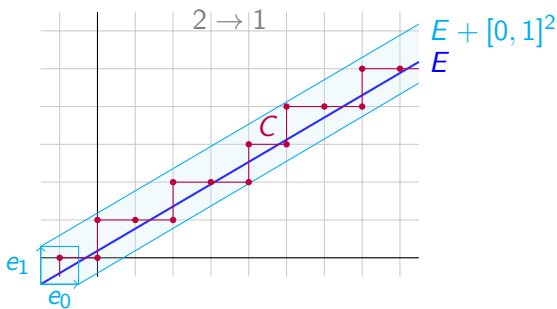


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# Cut and project $n \rightarrow d$

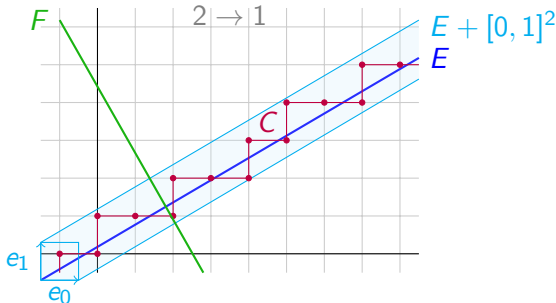
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**Projection**  $\pi : C \rightarrow E$  along  $F$  such that  $E \oplus F = \mathbb{R}^n$

We get a tiling of  $E$



# Cut and project $n \rightarrow d$

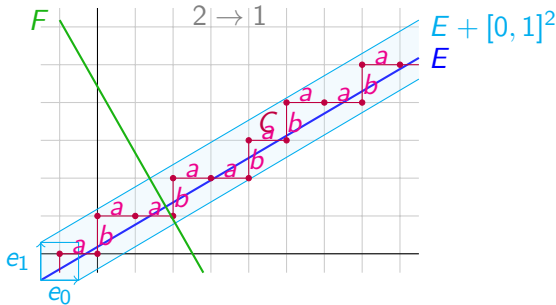
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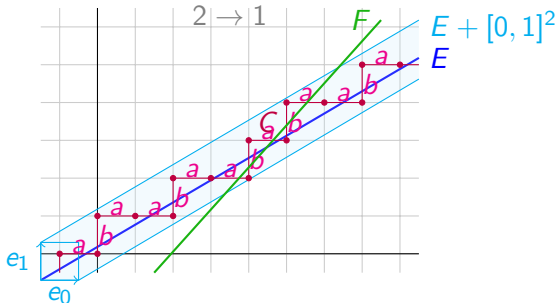
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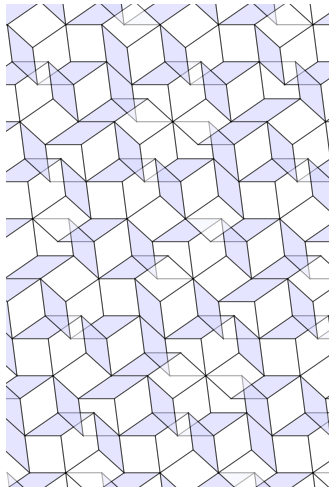
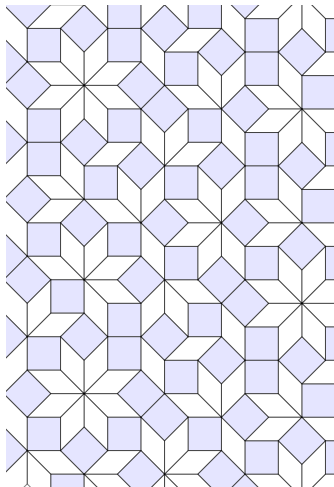
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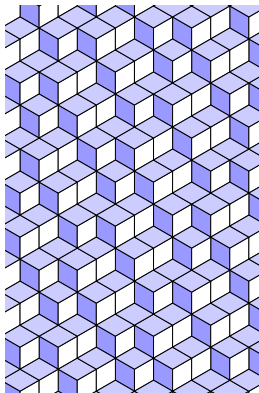
# Valid projection



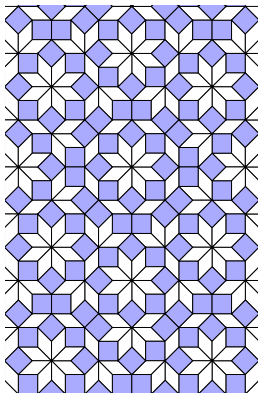
Valid  $\Leftrightarrow$  simple condition on determinants.

# Property and examples

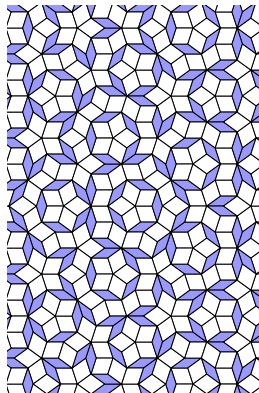
$E$  contains no rational line  $\Rightarrow$  non-periodic tilings



Rauzy  
 $3 \rightarrow 2$



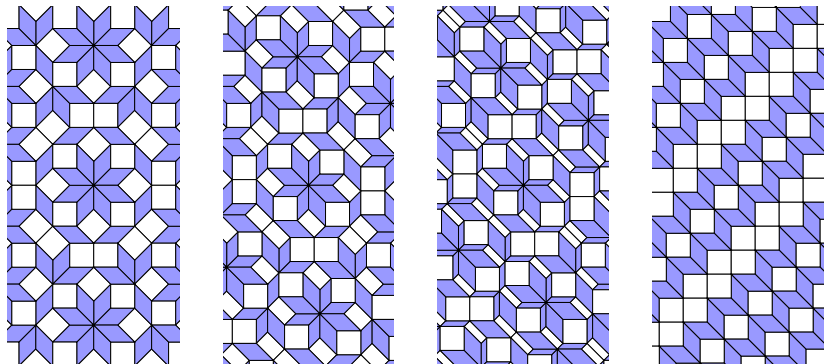
Ammann-Beenker  
 $4 \rightarrow 2$



Penrose  
 $5 \rightarrow 2$

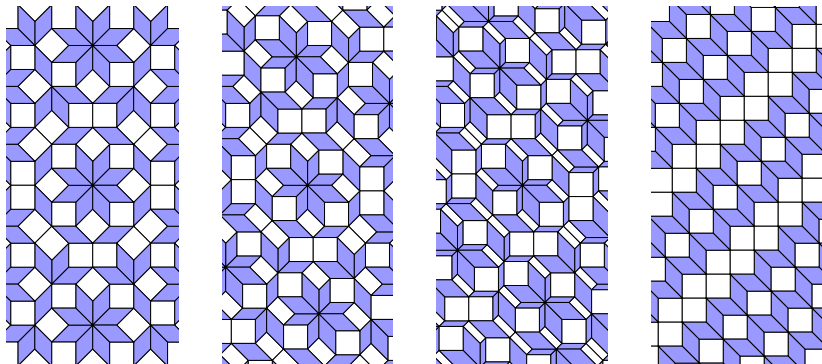
# Subperiods

**Shadow:** projection of the cut along  $n - d - 1$  base vectors.



# Subperiods

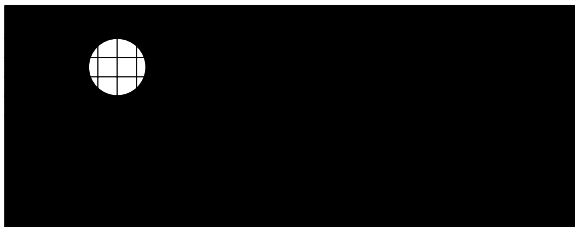
**Shadow:** projection of the cut along  $n - d - 1$  base vectors.



**Subperiod:** vector of  $E$  which projects on a period of a shadow.  
 It has  $d + 1$  integer coordinates.

## Local rules

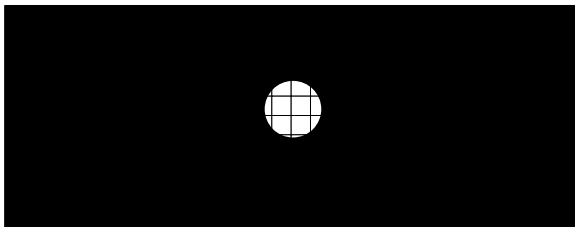
*Local rules:* patterns of a given size characterize a tiling.





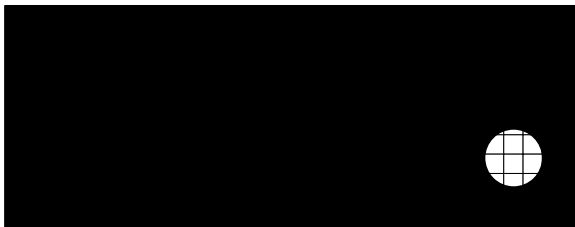
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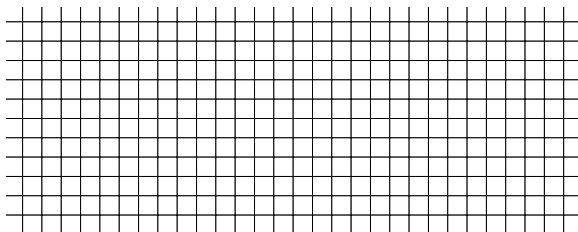
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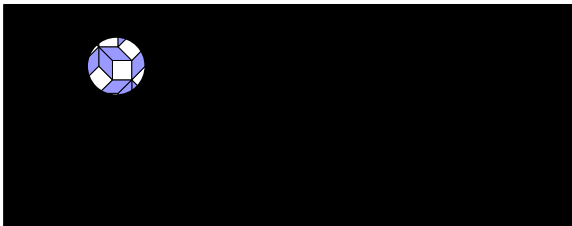
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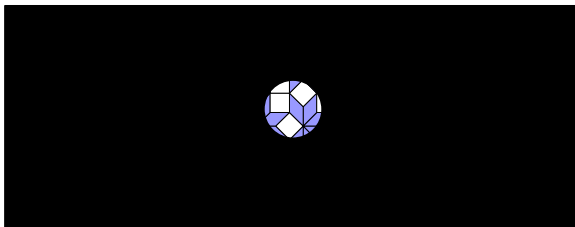
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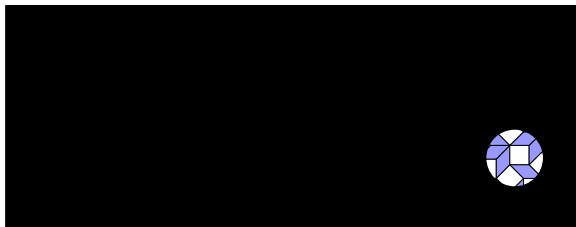
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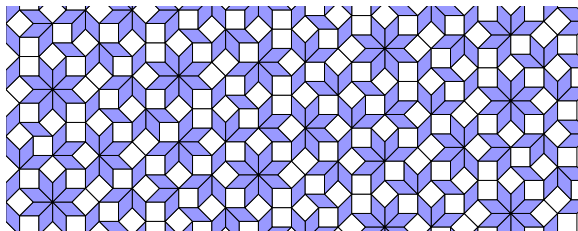
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# Local rules

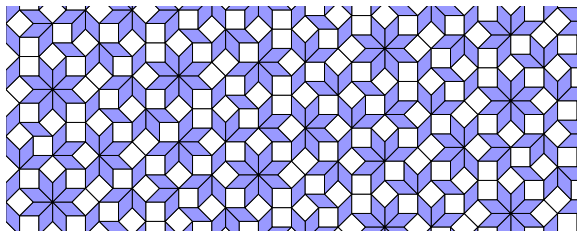
*Local rules:* patterns of a given size characterize a tiling.



Golden Octagonal tiling ( $4 \rightarrow 2$ )

# Local rules

*Local rules:* patterns of a given size characterize a tiling.



Golden Octagonal tiling ( $4 \rightarrow 2$ )

Theorem (Bédaride, Fernique, 2017)

*A 2-dim slope  $E \subset \mathbb{R}^4$  has local rules iff  $E$  is characterized by its subperiods.*



## FP-method to get Ammann bars

We give an algorithm which

- takes as input a 2-dim. slope  $E$  in  $\mathbb{R}^4$ ,
- checks whether  $E$  is characterized by subperiods,
- if so, checks whether  $E$  admits a *fine projection*,
- if so, outputs a finite set of tiles decorated by segments, such that Ammann bars rules yield  $4 \rightarrow 2$  tilings with slope  $E$ .

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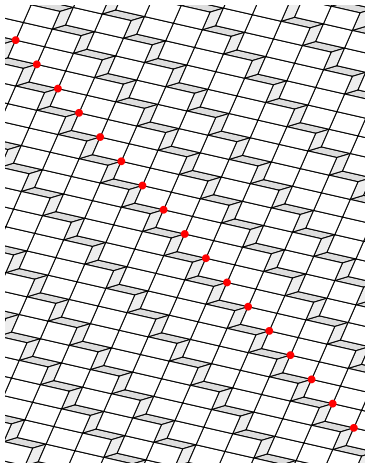
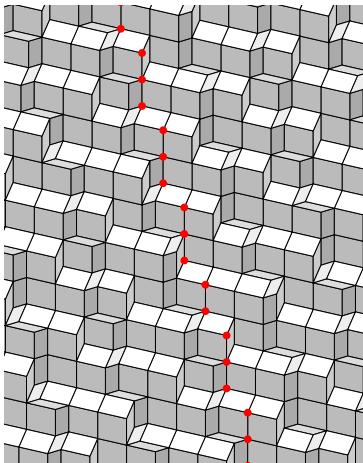
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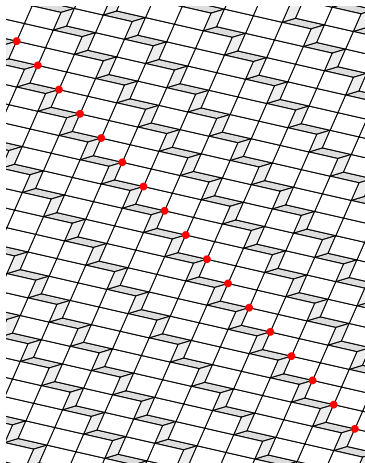
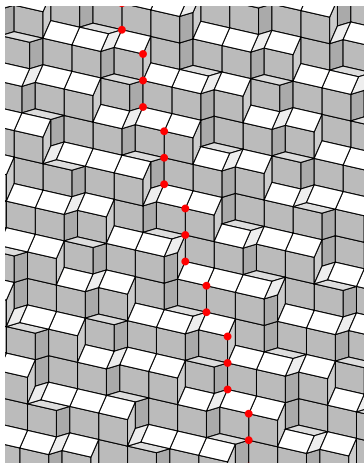
In short:

Local rules  $\simeq$  Subperiods  $\simeq$  Ammann bars

# Key idea

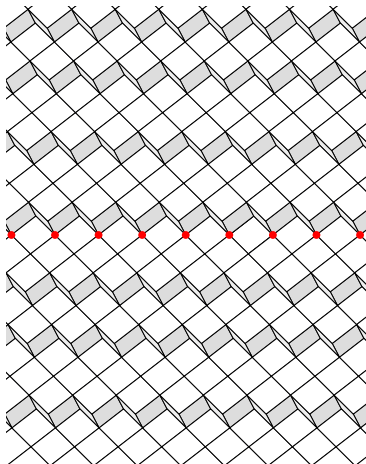
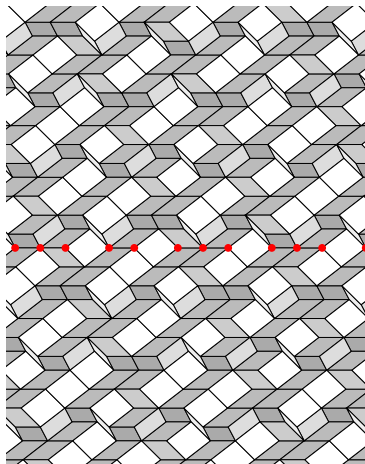


# Key idea



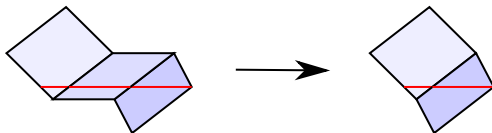
Choose a projection s.t. points which are aligned in the shadows are already aligned in the tiling, and force periodicity of the shadows using Ammann bars rules.

# Key idea



Choose a projection s.t. points which are aligned in the shadows are already aligned in the tiling, and force periodicity of the shadows using Ammann bars rules.

## Step 1: finding a fine projection

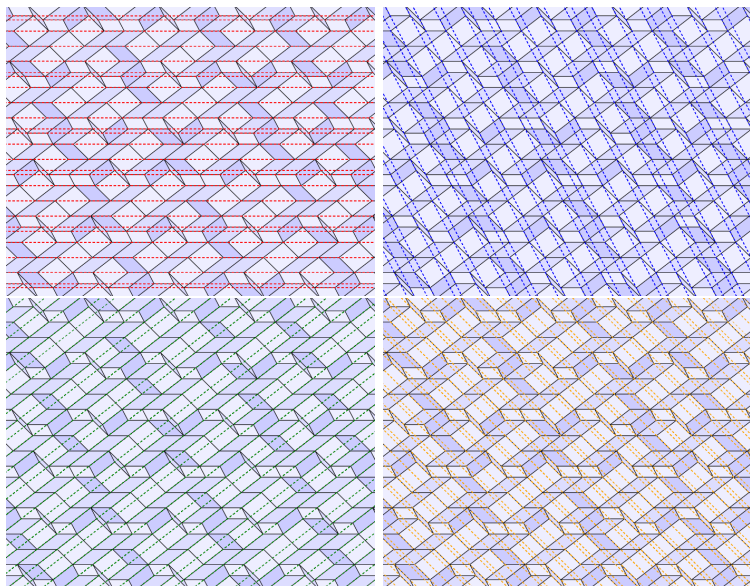


### Definition

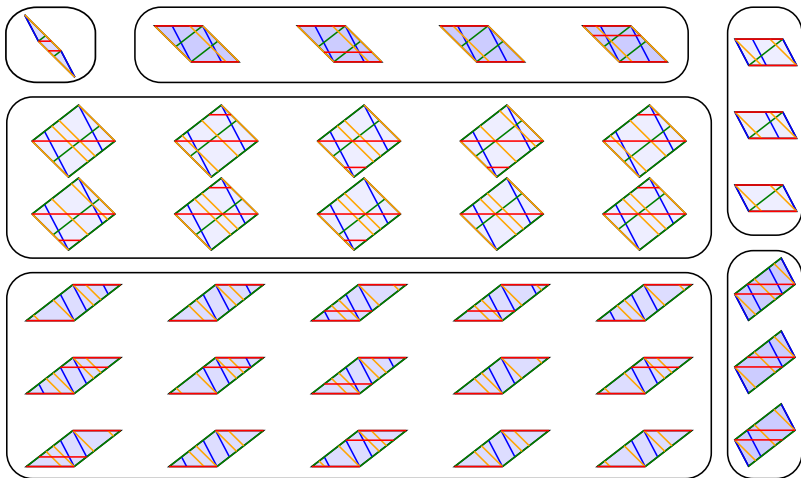
A **fine projection** for a 2-dimensional slope  $E \subset \mathbb{R}^4$  is a valid projection  $\pi : \mathbb{R}^4 \rightarrow \mathbb{R}^2$  such that for every  $i \in \{0, 1, 2, 3\}$ ,  $\pi(p_i)$  and  $\pi(e_i)$  are collinear.

- There is not always a fine projection: Golden Octagonal
- We provide an example for which it exists: Cyrenaic

## Step 2: drawing the lines directed by subperiods



# Step 3: generating a finite tileset

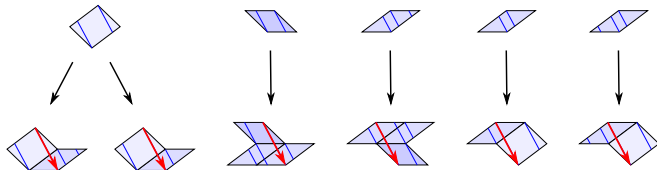




# Main results

## Theorem (Fernique, Porrier, 2022)

*Any set of tiles constructed by our algorithm from a 2-dim. slope  $E$  of  $\mathbb{R}^4$  forms tilings with the same subperiods as  $E$ .*



## Corollary

*If  $E$  is characterized by its subperiods then the tileset with Ammann bars rules yields only tilings with slope  $E$ .*

## Corollary

*If  $E$  is totally irrational then the tileset is aperiodic.*

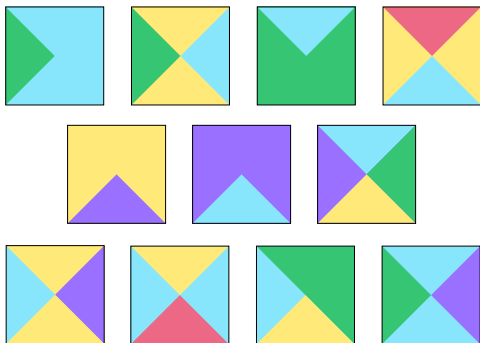
## Summary of Part 1

- New method to get aperiodic tilesets
- General result in "Ammann Bars for Octagonal Tilings":  
<https://arxiv.org/abs/2205.13973>
- Repo for our SageMath code:  
<https://github.com/cporrier/Cyrenaic>

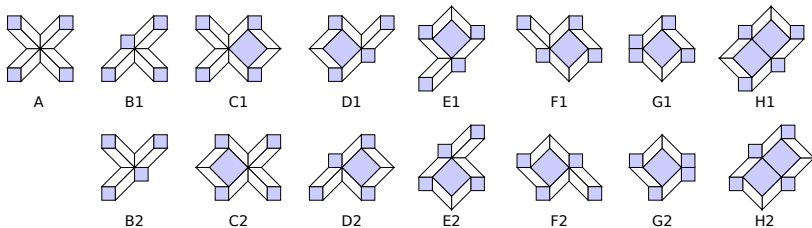
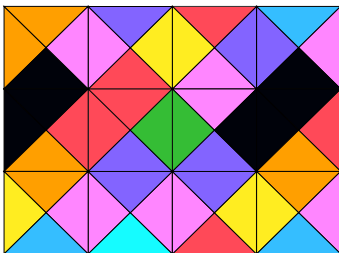
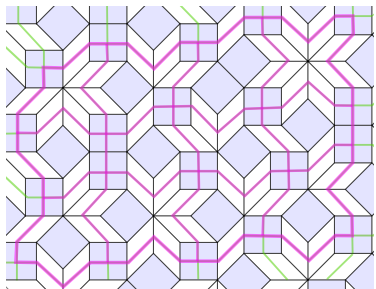
# Wang tiles

**Wang tiles** : unit squares of  $\mathbb{R}^2$  with colored edges; a tiling with Wang tiles is valid iff adjacent tiles have the same color on their common edge.

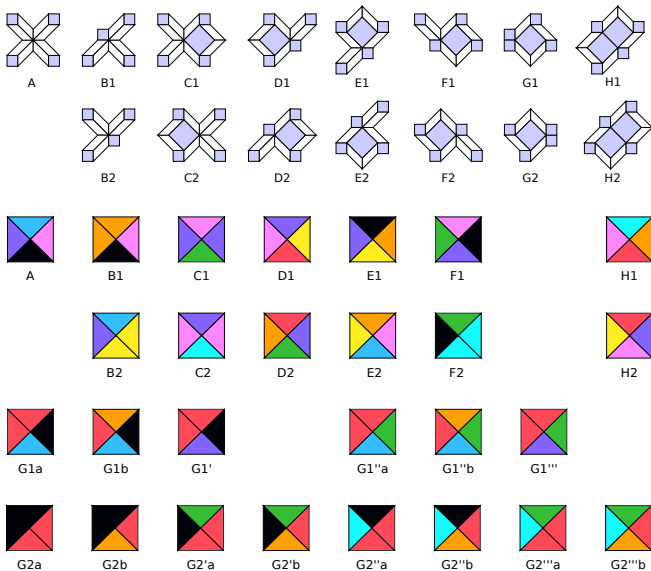
Example: Jeandel and Rao's set of 11 Wang tiles (smallest, 2015)



# Slicing of golden octagonal and patterns



# New aperiodic Wang tileset



# Cut and project $n \rightarrow d$

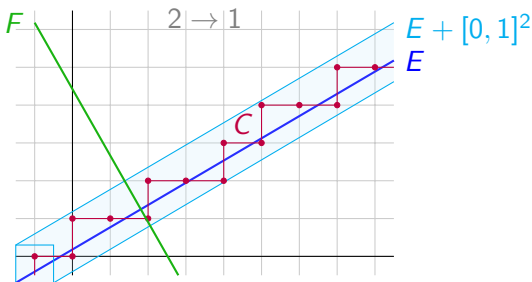
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$E \subset \mathbb{R}^n$  affine  $d$ -plane, called **slope**

**Cut**  $C : \{ d\text{-dim facets of } \mathbb{Z}^n, \subset E + [0, 1]^n \}$

**Projection**  $\pi : C \rightarrow E$  along  $F$  such that  $E \oplus F = \mathbb{R}^n$

We get a tiling of  $E$



# Cut and project $n \rightarrow d$

$e_0, \dots, e_{n-1}$  canonical basis of  $\mathbb{R}^n$

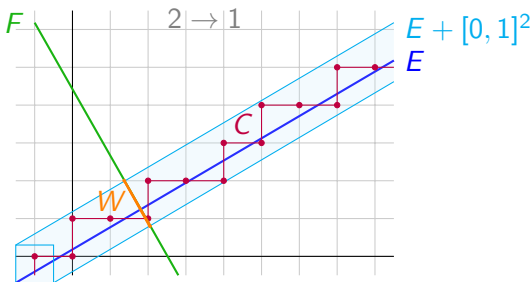
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**Window**  $W$ : image of  $E + [0, 1]^n$  projected on  $F$  along  $E$



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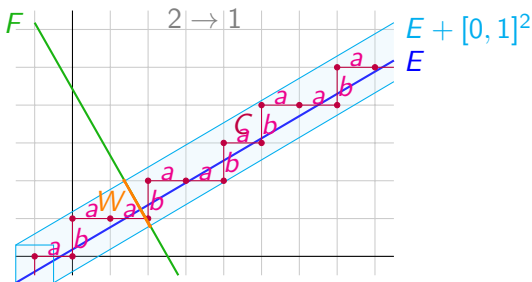
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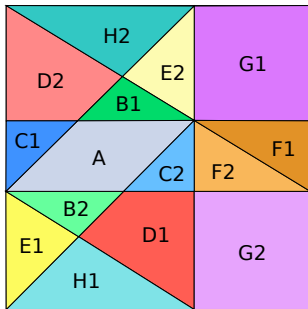
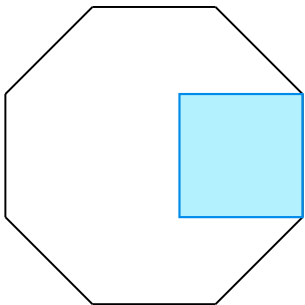
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# Frequencies of patterns

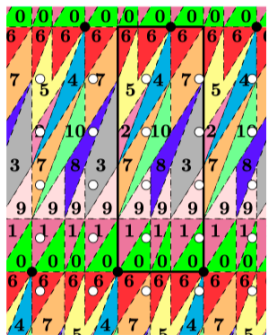


$$\begin{aligned} \nu([G1]) = \nu([G2]) &= 5 - 3\varphi \approx 14.5898\% \\ \nu([A]) = \nu([D1]) = \nu([D2]) &= 5\varphi - 8 \approx 9.017\% \\ \nu([H1]) = \nu([H2]) &= \frac{5}{2} - \frac{3}{2}\varphi \approx 7.2949\% \\ \nu([E1]) = \nu([E2]) = \nu([F1]) = \nu([F2]) &= \frac{5}{2}\varphi - 4 \approx 4.5085\% \\ \nu([B1]) = \nu([B2]) = \nu([C1]) = \nu([C2]) &= \frac{13}{2} - 4\varphi \approx 2.7864\% \end{aligned}$$

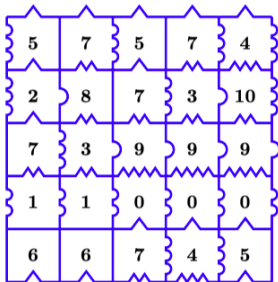
# Perspectives

Towards a symbolic representation for rotations on a torus.

Example of Labbé / Jeandel-Rao :  $\mathcal{D}_0 = (\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$  with  $\Gamma_0 = \langle (\varphi, 0), (1, \varphi + 3) \rangle_{\mathbb{Z}}$  and  $R_0(\mathbf{n}, \mathbf{x}) = R_0^n(\mathbf{x}) = \mathbf{x} + \mathbf{n}, \mathbf{n} \in \mathbb{Z}^2$



$$\begin{array}{c} \text{symb.} \\ \text{repr.} \end{array} \rightarrow \begin{pmatrix} 5 & 7 & 5 & 7 & 4 \\ 2 & 8 & 7 & 3 & 10 \\ 7 & 3 & 9 & 9 & 9 \\ 1 & 1 & 0 & 0 & 0 \\ 6 & 6 & 7 & 4 & 5 \end{pmatrix} \xrightarrow[\text{tiling}]{\text{Wang}}$$



Source: Sébastien Labbé, “Markov Partitions for Toral  $\mathbb{Z}^2$ -rotations Featuring Jeandel–Rao Wang Shift and Model Sets”

## References

- Baake, M. and Grimm, U. (2013). *Aperiodic Order. Vol 1. A Mathematical Invitation.*
- Bédaride, N. and Fernique, Th. (2015). When periodicities enforce aperiodicity.
- Grünbaum, B. and Shephard, G. C. (1987). *Tilings and Patterns.*
- Jeandel, E. and Rao, M. (2015). An aperiodic set of 11 Wang tiles.
- Labbé, S. (2021). Markov Partitions for Toral  $\mathbb{Z}^2$ -rotations Featuring Jeandel–Rao Wang Shift and Model Sets.
- Lind, D. and Marcus, B. (1995). *An Introduction to Symbolic Dynamics and Coding.*

Thank you for your attention!

4  $\rightarrow$  2 tilings of slope  $E$  characterized by subperiods:

$$p_0 = (\sqrt{3}, 0, 1, 1),$$

$$p_1 = (1, \sqrt{3} - 1, -1, 1),$$

$$p_2 = (1, -1, \sqrt{3} + 1, 0),$$

$$p_3 = (2, 1, -1, \sqrt{3}).$$

## Finding subperiods

Example: Golden octagonal tilings

Slope  $E$  generated by  $\vec{u} = (-1, 0, \varphi, \varphi)$  and  $\vec{v} = (0, 1, \varphi, 1)$ , where  $\varphi = \frac{1+\sqrt{5}}{2}$  is the golden mean.

Integer coordinates of  $p_0$ ?

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Integer coordinates of  $p_0$ ?  $p_0 = (*, 0, 1, 1)$

## Finding subperiods

Example: Golden octagonal tilings

Slope  $E$  generated by  $\vec{u} = (-1, 0, \varphi, \varphi)$  and  $\vec{v} = (0, 1, \varphi, 1)$ , where  $\varphi = \frac{1+\sqrt{5}}{2}$  is the golden mean.

Integer coordinates of  $p_0$ ?  $p_0 = (*, 0, 1, 1)$

Real coordinate?



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Integer coordinates of  $p_0$ ?  $p_0 = (*, 0, 1, 1)$

Real coordinate?  $p_0 = (1 - \varphi, 0, 1, 1)$