

# Discrepancy and nested perfect necklaces

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# Outline

Normality

Discrepancy

Nested perfect necklaces

Affine necklaces

## Normal sequences

A sequence of symbols is **normal** if all finite words of the same length occur in it with the same frequency.

If  $x \in A^{\mathbb{N}}$  and  $w \in A^*$ , the **frequency** of  $w$  in  $x$  is defined by

$$\text{freq}(x, w) = \lim_{N \rightarrow \infty} \frac{|x[1:N]|_w}{N}.$$

where  $|z|_w$  denotes the **number of occurrences** of  $w$  in  $z$ .

A sequence  $x \in A^{\mathbb{N}}$  is **normal** if for each  $w \in A^*$ :

$$\text{freq}(x, w) = \frac{1}{(\#A)^{|w|}}$$

where

- ▶  $\#A$  is the cardinality of the **alphabet**  $A$
- ▶  $|w|$  is the length of  $w$ .

## Normal sequences (continued)

Theorem (Borel, 1909)

*The decimal expansion of almost every real number in  $[0, 1)$  is a normal sequence of symbols in  $\{0, 1, \dots, 9\}$ .*

Nevertheless, not so many examples have been proved normal.  
Some of them are:

- ▶ Champernowne 1933 (natural numbers):

12345678910111213141516171819202122232425...

- ▶ Besicovitch 1935 (squares):

149162536496481100121144169196225256289324...

- ▶ Copeland and Erdős 1946 (primes):

235711131719232931374143475359616771737983...

# Equidistribution

A sequence  $(x_n)_{n \geq 1}$  of real numbers in  $[0; 1)$  is **equidistributed** if for any real numbers  $\alpha, \beta$  such that  $0 \leq \alpha \leq \beta \leq 1$ ,

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : x_n \in [\alpha; \beta]\}}{N} = \beta - \alpha$$

## Example

The sequence  $(\{n\alpha\})_{n \geq 1}$  is equidistributed for each irrational real number  $\alpha$  (where  $\{x\} = x - \lfloor x \rfloor$  is the fractional part of  $x$ ).

## Proposition

*The base  $b$  expansion of the real number  $\alpha$  is normal if and only if the sequence  $(\{b^n \alpha\})_{n \geq 1}$  is equidistributed.*

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The block  $w$  of length  $k$  occurs in  $\alpha$  at position  $n + 1$  iff  $\{b^n \alpha\}$  lies in the interval  $[0.w; 0.w + 2^{-k}]$ .

# Discrepancy

The **discrepancy**  $D_N$  of a sequence  $(x_n)_{n \geq 1}$  of real numbers in  $[0; 1)$  is defined by

$$D_N = \sup_{0 \leq \alpha \leq \beta \leq 1} \left| \frac{\#\{1 \leq n \leq N : x_n \in [\alpha; \beta]\}}{N} - (\beta - \alpha) \right|.$$

## Proposition

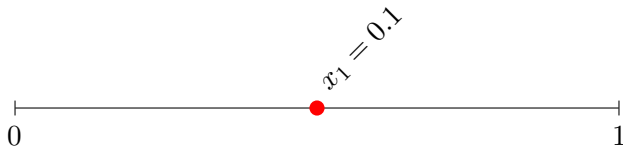
*The sequence  $(x_n)_{n \geq 1}$  is equidistributed iff  $\lim_{N \rightarrow \infty} D_N = 0$ .*

# Minimal discrepancy

## Theorem (Schmidt 72)

*There exists a constant  $C > 0$  such that for any sequence of real numbers, its discrepancy satisfies  $D_N \geq C \log N/N$  for infinitely many  $N \geq 1$ .*

This lower bound is achieved by the van der Corput sequence.



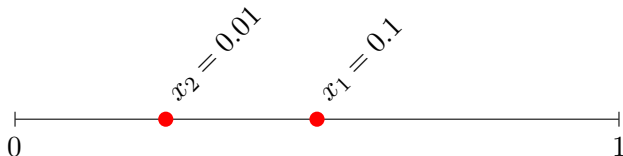


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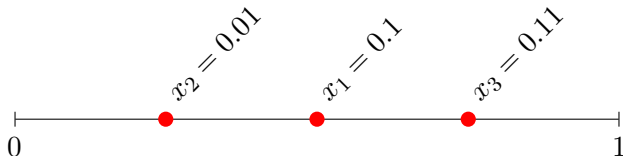


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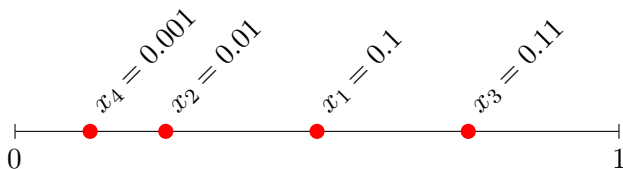


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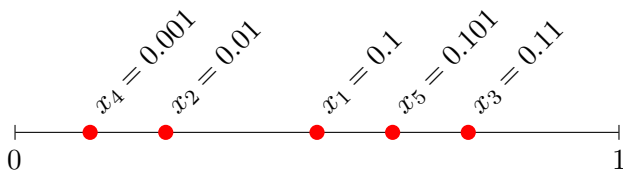


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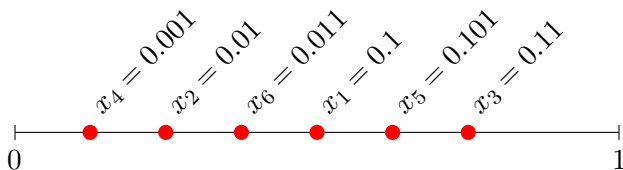


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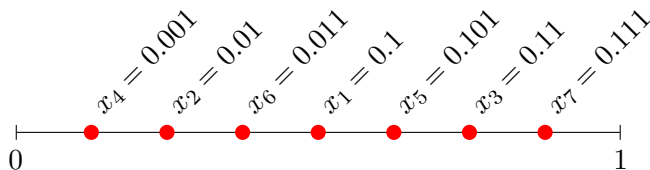


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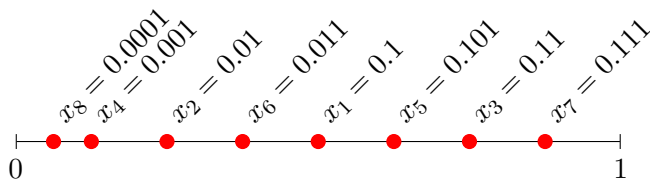


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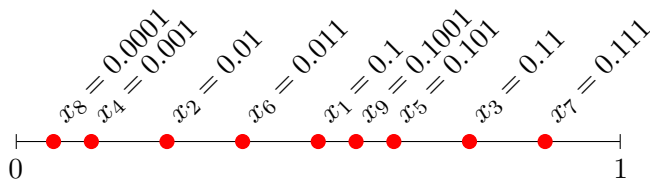


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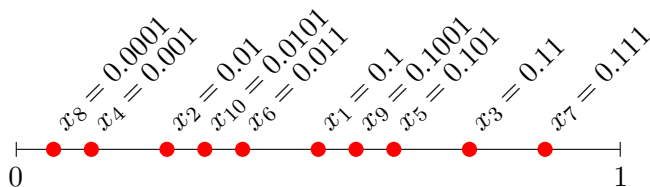


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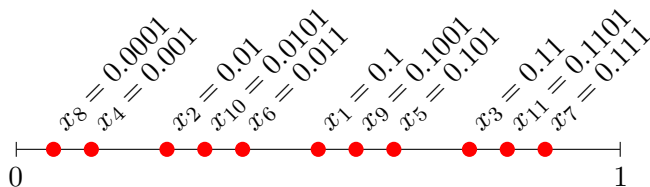


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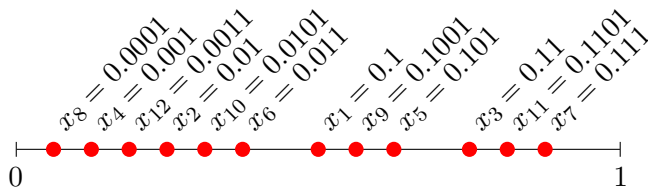


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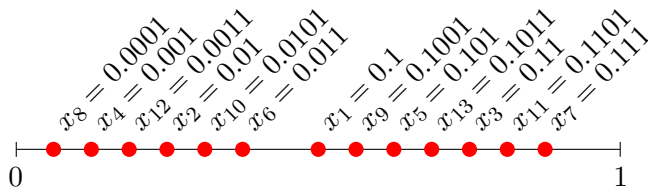


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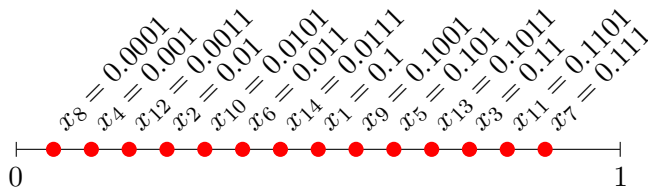


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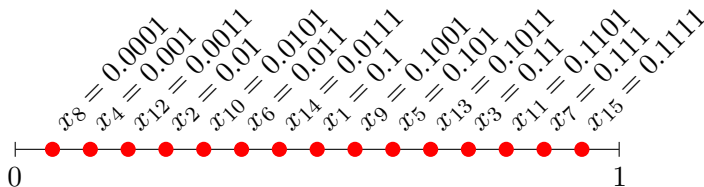


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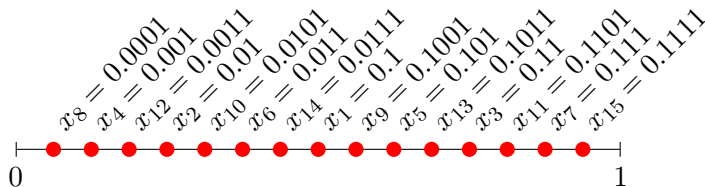


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For a sequence of the form  $(\{b^n \alpha\})_{n \geq 1}$ , the lowest discrepancy known so far is  $(\log N)^2/N$ . It is achieved by Levin's sequence.

## De Bruijn necklaces

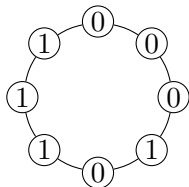
A **necklace** is a cyclically considered word (written on a circle).

$$\begin{array}{ccc} & 0 & \\ 1 & & 0 \\ & & & 0 \\ 1 & & & & 1 \\ & 0 & & & \end{array}$$



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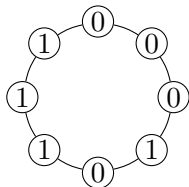


A  **$k$ -de Bruijn necklace** is a necklace in which each word of length  $k$  has exactly one (cyclic) occurrence. The necklace 00010111 is a 3-de Bruijn necklace.

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1		0	000	✓	100
1		0	001		101
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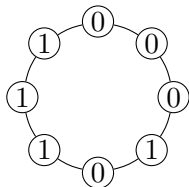


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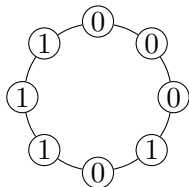


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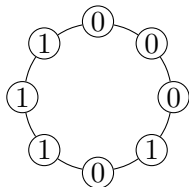


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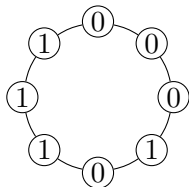


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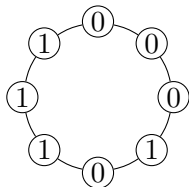


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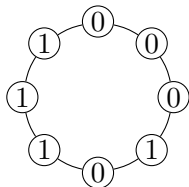


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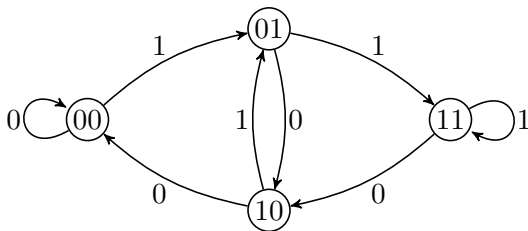
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## De Bruijn necklaces as Eulerian circuits

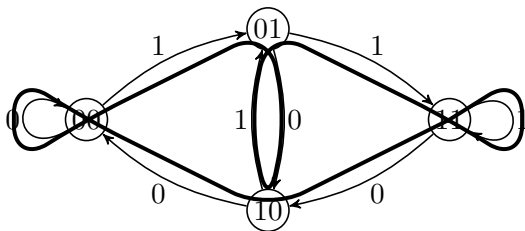
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The 3-de Bruijn necklace 00010111 corresponds to the following Eulerian circuit.



# BEST theorem

## Theorem (BEST 1951)

The number  $ec(G)$  of Eulerian circuits in a connected graph  $G = (V, E)$  is

$$ec(G) = t_w(G) \prod_{v \in V} (deg(v) - 1)!$$

where  $t_w(G)$  is the number of spanning trees directed to  $w$ . This number is equal to any cofactor of the Laplacian matrix.

## Proposition

The number of  $k$ -de Bruijn necklaces over an alphabet of size  $n$  is

$$\frac{(n!)^{n^{k-1}}}{n^k}.$$

- For  $n = 2$ ,
- ▶  $k = 1$  : 01 (and 10)
  - ▶  $k = 2$  : 0011 (0110, 1100 and 1001)
  - ▶  $k = 3$  : 00010111 and 11101000

## Perfect necklaces

A  $(k, m)$ -perfect necklace is a necklace in which each word of length  $k$  has  $m$  occurrences at positions which are different modulo  $m$ . Its length must be  $m(\#A)^k$ .

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The necklace 00010111 is not a  $(2, 2)$ -perfect necklace because the two occurrences of 01 have the same position modulo 2.

The necklace 00111001 is a  $(2, 2)$ -perfect necklace.

1	0	0	word	mod		word	mod
			00	1	✓	10	1
0		1	00	0		10	0
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## Perfect necklaces

A  $(k, m)$ -perfect necklace is a necklace in which each word of length  $k$  has  $m$  occurrences at positions which are different modulo  $m$ . Its length must be  $m(\#A)^k$ .

The necklace 000**1**0**1**11 is not a  $(2, 2)$ -perfect necklace because the two occurrences of 01 have the same position modulo 2.

The necklace 00111001 is a  $(2, 2)$ -perfect necklace.

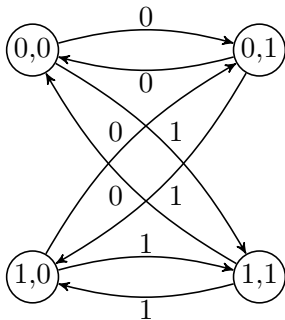
1	0		word	mod		word	mod	
		0	00	1	✓	10	1	✓
0			00	0	✓	10	0	✓
		1	01	1	✓	11	1	✓
0			01	0	✓	11	0	✓
	1							

More generally, the concatenation in lexicographic order of all words of length  $k$  gives a  $(k, k)$ -perfect necklace. Here is  $k = 3$ :

000001010011100101110111.

## Perfect necklaces as Eulerian circuits

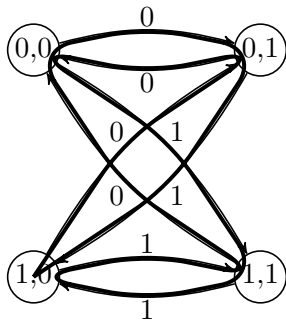
A  $(k, m)$ -perfect is an Eulerian circuit in the graph whose vertex set is  $A^{k-1} \times \{0, \dots, m-1\}$ .



## Perfect necklaces as Eulerian circuits

A  $(k, m)$ -perfect is an Eulerian circuit in the graph whose vertex set is  $A^{k-1} \times \{0, \dots, m-1\}$ .

The  $(2, 2)$ -perfect necklace 00111001 corresponds to the following Eulerian circuit.



## Nested perfect necklaces

A  $(k, m)$ -nested perfect necklace is a necklace such that for each  $1 \leq \ell \leq k$ , each block of length  $m(\#A)^\ell$  starting at a position multiple of  $m(\#A)^\ell$  is a  $(\ell, m)$ -perfect necklace. Its length is  $m(\#A)^k$ .

### Example

The necklace **00111001** is a  $(2, 2)$ -nested perfect necklace since

- ▶ **00111001** is a  $(2, 2)$ -perfect necklace
- ▶ **0011** and **1001** are  $(1, 2)$ -perfect necklaces



## Nested perfect necklaces

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### Example

0000111110100101110000110110100110000111001011010100101111100001

is a  $(4, 4)$ -nested perfect necklace since

- ▶ it is a  $(4, 4)$ -perfect necklace
- ▶ made of two  $(3, 4)$ -perfect necklaces

0000111110100101110000110110100110000111001011010100101111100001

- ▶ made of four  $(2, 4)$ -perfect necklaces

0000111110100101110000110110100110000111001011010100101111100001

- ▶ made of eight  $(1, 4)$ -perfect necklaces

0000111110100101110000110110100110000111001011010100101111100001

# Levin's sequence

## Proposition

Levin's sequence of discrepancy  $(\log N)^2/N$  is a concatenation  $u_0u_1u_2\cdots$  where each  $u_n$  is a  $(2^n, 2^n)$ -nested perfect necklace of length  $2^{n+2^n}$ :

$$\underbrace{01}_{2} \underbrace{00111001}_{8} \underbrace{000011111010\dots}_{64} \underbrace{0000000011111111010\dots}_{2048}$$

## Levin's construction

Let  $n \geq 0$ . Let  $\hat{M}_n$  be the  $2^n \times 2^n$ -matrix defined by  $M_{1,i} = M_{i,2^n} = 1$  for  $1 \leq i \leq 2^n$  and  $M_{i,j} = M_{i-1,j} + M_{i,j+1}$ .

$$\hat{M}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 \\ 10 & 6 & 3 & 1 \\ 20 & 10 & 4 & 1 \end{pmatrix}$$

Let  $M_n$  be the matrix over  $\mathbb{F}_2$  defined by  $M_n \equiv \hat{M}_n \pmod{2}$ . The matrices  $M_n$  can also be defined recursively by

$$M_0 = (1) \quad \text{and} \quad M_{n+1} = \begin{pmatrix} M_n & M_n \\ 0 & M_n \end{pmatrix}.$$

The matrices  $M_1$  and  $M_2$  are then

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

## Levin's construction (continued)

Words over  $\{0, 1\}$  are identified with column vectors over  $\mathbb{F}_2$ . Let  $w_1, \dots, w_{2^{2^n}}$  be the enumeration in lexicographic order of all words of length  $2^n$  over  $\{0, 1\}$ . Let  $w'_i = M_n w_i$  for each  $1 \leq i \leq 2^{2^n}$ .

### Theorem

*The concatenation  $w'_1 \cdots w'_{2^{2^n}}$  is a  $(2^n, 2^n)$ -nested perfect necklace.*

### Examples

For  $n = 1$ ,

$i$	1	2	3	4
$w_i$	00	01	10	11
$w'_i$	00	11	10	01

For  $n = 2$ ,

$i$	1	2	$\dots$	16
$w_i$	0000	0001	$\dots$	1111
$w'_i$	0000	1111	$\dots$	0001

## More matrices

Let  $k_1, \dots, k_m$  be integers such that  $k_i - 1 \leq k_{i+1} \leq k_i$  for  $1 \leq i < 2^n$  and  $k_{2^n} = 0$ . Define

$$M_n^{k_1, \dots, k_{2^n}} = (\sigma^{k_1}(C_1), \dots, \sigma^{k_m}(C_m)).$$

where  $M_n = (C_1, \dots, C_m)$ . For  $n = 2$ , the eight matrices:

$$\begin{array}{cccc} M_2^{0,0,0,0} & M_2^{1,0,0,0} & M_2^{1,1,0,0} & M_2^{2,1,0,0} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{cccc} M_2^{1,1,1,0} & M_2^{2,1,1,0} & M_2^{2,2,1,0} & M_2^{3,2,1,0} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

## Affine necklaces

Let  $M$  be one matrix  $M_n^{k_1, \dots, k_{2^n}}$  and  $z$  be a fixed vector of length  $2^n$ . Let  $w'_i = Mw_i + z$  for each  $1 \leq i \leq 2^{2^n}$ . An **affine necklace** is the concatenation  $w'_1 \cdots w'_{2^n}$ .

### Theorem

*A necklace is a nested perfect necklace if and only if it is an affine necklace. There are  $2^{2^{n+1}-1}$   $(2^n, 2^n)$ -nested perfect necklaces.*

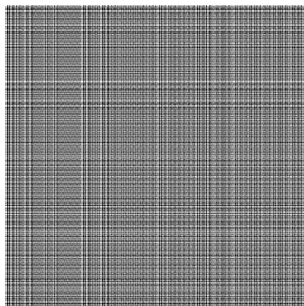
### Theorem

*Any sequence which is the concatenation of  $(2^n, 2^n)$ -nested perfect necklaces for  $n = 0, 1, 2, \dots$  has discrepancy  $(\log N)^2/N$  (as the one defined by Levin).*

## Open questions

- ▶ What is the number of  $(k, m)$ -nested perfect necklaces over some alphabet  $A$  ? We only know when  $\#A = 2$  and  $m$  is a power of 2.
- ▶ What is the least possible discrepancy for a sequence of the form  $(\{b^n \alpha\})_{n \geq 1}$  for some real number  $\alpha$  ?
- ▶ 2-dimensional objects : nested perfect toroidal arrays

```
0 0 0 1 0 1 0 0
0 0 0 1 0 0 0 1
1 0 1 1 1 1 1 0
1 0 1 1 1 0 1 1
1 0 1 1 1 1 1 0
0 0 0 1 0 0 0 1
0 0 0 1 0 1 0 0
1 0 1 1 1 0 1 1
```



Merci