The canonical complex of the weak order

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GT Combinatoire et interactions

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Joint work with Vincent Pilaud (CNRS & École Polytechnique)
Lattices and canonical join representations
Definition

A (finite) **lattice** $L$ is a (finite) poset where every family $X$ of elements of $L$ has a **join** $\bigvee X$ (smallest upper bound) and a **meet** $\bigwedge X$ (greatest lower bound).

Definition

The **canonical join representation** of an element $x$ is a subset $J \subseteq L$ such that:

- $\bigvee J = x$,
- $J' \subsetneq J \Rightarrow \bigvee J' \neq x$,
- $J$ is **lowest** in $L$ with these properties.

When it always exists, we call the lattice **join semidistributive**.
Irreducibility

Definition
The elements that are their own canonical join representation are the join irreducibles. In finite lattices, they are those covering only one element. Canonical join representations are made of join irreducibles.
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LaBRI (UB) 3 / 29
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**Proposition (Freese Nation ’95)**

\[
cjr(x) = \{ k_\vee(x, y) \mid y \leq x \},
\]

where \( k_\vee(x, y) := \min\{z \leq x, z \not\leq y\} \).
Proposition (Freese Nation ’95)

\[ \text{cjr}(x) = \{ k \lor (x, y) \mid y \preceq x \}, \]

where \( k \lor (x, y) := \min\{z \leq x, z \not\preceq y\} \).
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\[ k \lor (x, a \lor b) = c. \]
Proposition (Freese Nation’95)

\[ \text{cjr}(x) = \{ k_\vee(x, y) \mid y \lessdot x \}, \]

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\[ k \lor (x, b \lor c) = a. \]
\[ \text{cjr}(x) = \{ a, b, c \}. \]
Definition (Reading ’15, Barnard ’19, ’20)

The \textbf{canonical join complex} associated to a join semidistributive lattice $L$ is the simplicial complex $CJC(L)$ with:

- \textbf{vertices} := \{join irreducibles\},
- \textbf{faces} := \{canonical join representations\}.
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- vertices $\{\text{join irreducibles}\}$,
- faces $\{\text{canonical join representations}\}$.

Theorem (Reading ’15)

*It is a flag simplicial complex.*
Lattices and canonical join representations

Weak order on permutations and arcs

Canonical complexes

Canonical join complex

The canonical complex of the weak order
Lattices and canonical join representations

Weak order on permutations and arcs

Canonical complexes

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Lattice $L$

Canonical join complex $\mathcal{JC}(L)$
Lattices and canonical join representations

Weak order on permutations and arcs

Canonical complexes

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Lattice $L$

Canonical join complex $\mathcal{CJC}(L)$
Definition

A lattice congruence is an equivalence relation $\equiv$ on $L$ such that $x \equiv x'$ and $y \equiv y'$ implies $x \lor y \equiv x' \lor y'$ and $x \land y \equiv x' \land y'$.
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Weak order on permutations and arcs

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Theorem (Reading ’16)

A lattice congruence is characterized by the join irreducibles it contracts (merge with the one they cover). More precisely, there is a poset on join irreducibles called forcing order such that all ideals of this poset correspond to a lattice congruence.
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Theorem (Reading ’15)
The canonical join complex behaves well with lattice congruences.
Lattices and canonical join representations
Weak order on permutations and arcs
Canonical complexes

Lattice congruences

The canonical complex of the weak order

LaBRI (UB) 9 / 29
Lattices and canonical join representations

Weak order on permutations and arcs

Canonical complexes

Lattice congruences

\[ \begin{array}{c}
L \\
\mathrel{\lor} \emptyset \\
\mathrel{\lor} b \\
\mathrel{\lor} c \\
\mathrel{\lor} a \\
\mathrel{\lor} d \\
\mathrel{\lor} e \\
\mathrel{\lor} b \lor c \\
\mathrel{\lor} a \lor c \\
\mathrel{\lor} a \lor b \\
\mathrel{\lor} a \\
\end{array} \]

\[ CJC(L) \]

Forcing order

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The canonical complex of the weak order

LaBRI (UB) 9 / 29
Forcing order

Ideal

\( CJC(L) \)
Lattices and canonical join representations

Weak order on permutations and arcs

Canonical complexes

Lattice congruences

Forcing order

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Forcing order

\( CJC(L) \)
Lattices and canonical join representations

Weak order on permutations and arcs

Canonical complexes

Lattice congruences

The canonical complex of the weak order

LaBRI (UB) 9 / 29
Lattices and canonical join representations

Weak order on permutations and arcs

Canonical complexes

Lattice congruences

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The canonical complex of the weak order
Lattices and canonical join representations

Weak order on permutations and arcs

Canonical complexes

Lattice congruences

The canonical complex of the weak order

LaBRI (UB) 9 / 29
Lattices and canonical join representations

Weak order on permutations and arcs

Canonical complexes

Lattice congruences

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The canonical complex of the weak order

LaBRI (UB)
Lattices and canonical join representations

Weak order on permutations and arcs

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CJC(L)

CJC(L/≡)

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The canonical complex of the weak order

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Weak order on permutations and arcs
Proposition

The (right) weak order is a semidistributive lattice on permutations ordered by containment of their inversion sets.

\[ \text{inv}(132) = \{(2, 3)\} \subseteq \{(1, 3), (2, 3)\} = \text{inv}(312) \]

\[ 132 \preceq 312 \]
A nice bijection

$$\sigma = 526413$$
A nice bijection

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**Permutation table:**
\[ \{(\sigma_i, i) \mid i \in [n]\} \].
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**Theorem (Reading '15)**

This is a bijection between permutations and Non-Crossing Arc Diagrams (NCADs).
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*Flatten!*
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**The canonical complex of the weak order**

LaBRI (UB) 12 / 29
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**Theorem (Reading ’15)**

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The bijection between permutations and NCADs provides a combinatorial model for the canonical join representations in the weak order.
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\[ \sigma \leftrightarrow \text{cjr(\(\sigma\))} \]

join irreducible permutations  \[\sigma \vee\]  single arcs
Theorem (Reading ’15)

The canonical join complex of the weak order is isomorphic to the non-crossing complex.
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Proposition (Reading ’15)

The forcing on arcs corresponds to the extension of arcs.
Canonical complexes
And what about meet?

Time well spent

Everything we said has a counterpart in terms of canonical meet representations, canonical meet complexes and NCADs.
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The $\kappa$ maps

\[
\begin{array}{c}
\begin{array}{c}
c \downarrow \quad b \downarrow \\
\quad \quad a \\
d \downarrow \\
e
\end{array} \\
\end{array}
\]

$\mathcal{JC}(L)$

\[
\begin{array}{c}
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g \uparrow \\
h \uparrow \\
e \uparrow \\
f \uparrow \\
d \uparrow \\
\end{array} \\
\end{array}
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$\mathcal{MC}(L)$

Theorem (A., Pilaud '22+)

The $\kappa$ maps are simplicial complex isomorphism.

For $L = S_n$, the $\kappa$ maps correspond to changing the color of the arc.
The $\kappa$ maps

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- The $\kappa$ maps are simplicial complex isomorphism.
- For $L = S_n$, the $\kappa$ maps correspond to changing the color of the arc.
Definition (A., Pilaud '22+)

**Canonical representation** of an interval:

\[ \text{cr}([x, y]) := \text{cjr}(x) \sqcup \text{cmr}(y). \]

**Canonical complex** \( CC(L) \) of a semidistributive lattice \( L \):

- vertices := \{join irreducibles\} \sqcup \{meet irreducibles\},
- faces := \( J \sqcup M \) such that:
  - \( J \) is a canonical join representation,
  - \( M \) is a canonical meet representation,
  - \( \bigvee J \leq \bigwedge M \).

So that the faces of the canonical complex \( CC(L) \) are in bijection with the interval of \( L \).
The canonical complex ...

... is a simplicial complex, i.e., one can embed it on the boundary of a simplex.
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**Theorem (A., Pilaud ’22+)**

*The canonical complex behaves as well as the canonical join and meet complexes with respect to taking quotients of the lattice.*
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\textbf{Theorem (A., Pilaud '22+)}

\textit{The canonical complex behaves as well as the canonical join and meet complexes with respect to taking quotients of the lattice. The canonical complex of a quotient is the subcomplex of the canonical complex spanned by the uncontracted join or meet irreducibles.}
Back to our example

The canonical complex of the weak order
Back to our example

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Back to our example

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The canonical complex of the weak order
Back to our example

Lattices and canonical join representations
Weak order on permutations and arcs
Canonical complexes

The canonical complex of the weak order

LaBRI (UB) 21 / 29
Back to our example

The canonical complex of the weak order

LaBRI (UB) 21 / 29
Back to our example

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The canonical complex of the weak order
Back to our example
Back to our example

The canonical complex of the weak order

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With the interval $[526413, 564231]$, we associate the superimposition of diagrams:
Theorem (A., Pilaud ’22+)

This is a bijection between intervals of the weak order and Semi-Crossing Arc Bidiagrams (SCABs).
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This is a bijection between intervals of the weak order and Semi-Crossing Arc Bidiagrams (SCABs).
Theorem (A., Pilaud '22+)

This bijection between intervals of the weak order and SCABs provides a combinatorial model for the canonical complex of the weak order: the semi-crossing complex.
\[ \alpha = (a, b, A, B) = (3, 8, \{6, 7\}, \{4, 5\}) \]
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**Lemma**

For any two arcs \( \alpha := (a, b, A, B) \) and \( \alpha' := (a', b', A', B') \), then

1. \( \sigma_\vee(\alpha) \leq \sigma_\vee(\alpha') \) ⇔

2. \( \sigma_\wedge(\alpha) \leq \sigma_\wedge(\alpha') \) ⇔

3. \( \sigma_\vee(\alpha) \leq \sigma_\wedge(\alpha') \) ⇔
Lemma

For any two arcs $\alpha := (a, b, A, B)$ and $\alpha' := (a', b', A', B')$, then

- $\sigma \vee (\alpha) \leq \sigma \vee (\alpha') \iff a \in B' \cup \{a\}$ and $b \in A' \cup \{b\}$, and $A \subseteq A'$ and $B \subseteq B'$,

- $\sigma \wedge (\alpha) \leq \sigma \wedge (\alpha') \iff a' \in B \cup \{a\}$ and $b' \in A \cup \{b\}$, and $A' \subseteq A$ and $B' \subseteq B$,

- $\sigma \vee (\alpha) \leq \sigma \wedge (\alpha') \iff \exists u < v$, $u \in (A' \cup \{a\}) \cap (B \cup \{a\})$ and $v \in (A \cup \{b\}) \cap (B' \cup \{b\})$.

\[
\alpha = (a, b, A, B) = (3, 8, \{6, 7\}, \{4, 5\})
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Lemma

For any two arcs $\alpha := (a, b, A, B)$ and $\alpha' := (a', b', A', B')$, then

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2. $\sigma_\wedge(\alpha) \leq \sigma_\wedge(\alpha') \iff a' \in B \cup \{a\}$ and $b' \in A \cup \{b\}$, and $A' \subseteq A$ and $B' \subseteq B$,

3. $\sigma_v(\alpha) \leq \sigma_\wedge(\alpha') \iff \not \exists u < v$, $u \in (A' \cup \{a'\}) \cap (B \cup \{a\})$ and $v \in (A \cup \{b\}) \cap (B' \cup \{b'\})$. 

Where:

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- \( \sigma \wedge (\alpha) \leq \sigma \wedge (\alpha') \iff a' \in B \cup \{a\} \) and \( b' \in A \cup \{b\} \), and \( A' \subseteq A \) and \( B' \subseteq B \),

- \( \sigma \vee (\alpha) \leq \sigma \wedge (\alpha') \iff \not\exists u < v, u \in (A' \cup \{a'\}) \cap (B \cup \{a\}) \) and \( v \in (A \cup \{b\}) \cap (B' \cup \{b'\}) \).
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**Lemma**

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- \( \sigma_\wedge(\alpha) \leq \sigma_\wedge(\alpha') \iff a' \in B \cup \{a\} \) and \( b' \in A \cup \{b\} \), and \( A' \subseteq A \) and \( B' \subseteq B \),
- \( \sigma_\vee(\alpha) \leq \sigma_\wedge(\alpha') \iff \nexists u < v, u \in (A' \cup \{a'\}) \cap (B \cup \{a\}) \) and \( v \in (A \cup \{b\}) \cap (B' \cup \{b'\}) \).
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**Lemma**

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- \( \sigma \vee (\alpha) \leq \sigma \vee (\alpha') \iff a \in B' \cup \{a'\} \) and \( b \in A' \cup \{b'\} \), and \( A \subseteq A' \) and \( B \subseteq B' \),
- \( \sigma \wedge (\alpha) \leq \sigma \wedge (\alpha') \iff a' \in B \cup \{a\} \) and \( b' \in A \cup \{b\} \), and \( A' \subseteq A \) and \( B' \subseteq B \),
- \( \sigma \vee (\alpha) \leq \sigma \wedge (\alpha') \iff \exists u < v, u \in (A' \cup \{a'\}) \cap (B \cup \{a\}) \) and \( v \in (A \cup \{b\}) \cap (B' \cup \{b'\}) \).
\( \alpha = (a, b, A, B) = (3, 8, \{6, 7\}, \{4, 5\}) \)

**Lemma**

*For any two arcs \( \alpha := (a, b, A, B) \) and \( \alpha' := (a', b', A', B') \), then*

- \( \sigma \lor (\alpha) \leq \sigma \lor (\alpha') \iff a \in B' \cup \{a'\} \) and \( b \in A' \cup \{b'\} \), and \( A \subseteq A' \) and \( B \subseteq B' \),

- \( \sigma \land (\alpha) \leq \sigma \land (\alpha') \iff a' \in B \cup \{a\} \) and \( b' \in A \cup \{b\} \), and \( A' \subseteq A \) and \( B' \subseteq B \),

- \( \sigma \lor (\alpha) \leq \sigma \land (\alpha') \iff \nexists u < v, u \in (A' \cup \{a'\}) \cap (B \cup \{a\}) \) and \( v \in (A \cup \{b\}) \cap (B' \cup \{b'\}) \).
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Some SCABs

\[ [1234, 4321] \]
[1234, 4321]
[1234, 1234]
Some SCABs

[2314, 2314]
[2413, 4321]
Some SCABs

\[ \{2413, 4321\} \]
Some SCABs

\[
[??? , ???]
\]
[1324, 3412]
Some SCABs

\[[2143, 2413]\]
Some SCABs
Algorithm

Given a congruence $\equiv$ of the weak order and the canonical meet representation of the top element of a class, find the canonical join representation of the bottom element of this class.
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When $\equiv$ is the sylvester congruence, i.e., when it contracts all arcs but those shaped like $\bullet$, we recover the classical Kreweras complement on non-crossing partitions:
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Geometry (motivation):

- Building a Hopf Algebra on the faces of quotientopes consistent with previous constructions (Chapoton '00, Aguiar Ardila '17, Pilaud '19).
- Given a congruence, what are the SCABs that correspond to a face of the associated quotientope? (solved only for the trivial congruence)
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Generalizations:
- Can we generalize the canonical complex to the study of $k$-chains?
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• **Generalizations:**
  - Can we generalize the canonical complex to the study of $k$-chains?

• **Study of the canonical complex:**
  - What are the symmetries of the canonical complex in general?
  - If we add structure on the lattice, do we get structure on its canonical complex? We studied the distributive case, what about congruence uniformity?
  - What can we guess of the shape of an interval given the shape of its SCAB? (question from Sara Billey)
Thank you!
Take $n = 4$ and $\equiv$ contracting only $\bullet \bullet \bullet$. 
• Take \( n = 4 \) and \( \equiv \) contracting only

• We start with the diagram \( \bullet \bullet \bullet \).
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Take the associated join diagram: $\bullet \bullet \bullet \bullet$. 
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Take the associated join diagram:

Consider all the set of arcs beneath any arc of this diagram in the weak order.
Take \( n = 4 \) and \( \equiv \) contracting only \( \bullet \bullet \bullet \bullet \).

We start with the diagram \( \bullet \bullet \bullet \bullet \).

Take the associated join diagram: \( \bullet \bullet \bullet \bullet \).

Consider all the set of arcs beneath any arc of this diagram in the weak order.

Remove those that are contracted by \( \equiv \).

Remove those that are a fusion of two others.
Take $n = 4$ and $\equiv$ contracting only.

We start with the diagram.

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Remove those that are a fusion of two others.

Take the maximal elements.

is a SCAB corresponding to a class of $\equiv$!
Example of Kreweras complement
The canonical complex of the weak order

$$\kappa_{\land}(b) := \max \{x \geq \bigvee \emptyset, x \nleq b\}$$
\[ \kappa_{\land}(b) := \max\{x \geq \bigvee \emptyset, x \not\leq b\} \]
Lattices and canonical join representations

Weak order on permutations and arcs

Canonical complexes

\( \kappa \) computations

The canonical complex of the weak order

LaBRI (UB) 32 / 29

\[ \kappa_\wedge (b) := \max\{x \geq \bigvee \emptyset, x \nleq b\} \]
\[ \kappa_\wedge(b) := \max\{x \geq \bigvee \emptyset, x \not\geq b\} \]
\[ \kappa_\wedge(b) := \max\{x \geq \bigvee \emptyset, x \not\geq b\} = h \]
These maps were studied in Barnard '19, extended in Defant-Williams '22.