

Density of sphere packings: from coins to oranges

Daria Pchelina














LIPN, CALIN

supervised by

Thomas Fernique

20/3/2023

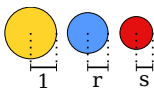
Plan

- 1 What is a packing?
- 2 Packing  in 2D and  in 3D
- 3 Triangulated packings
- 4 Proof for 
- 5 Proof for   
- 6 Proof strategies for ,    and 
- 7 Work in progress:  

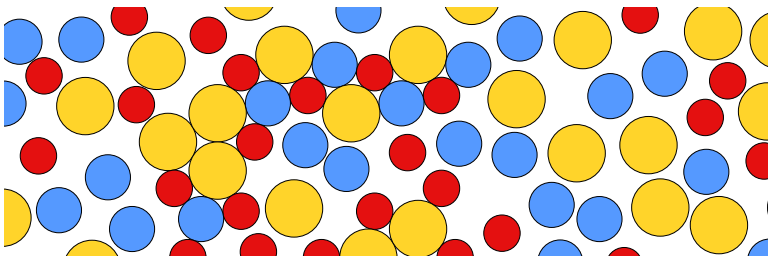


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Discs:

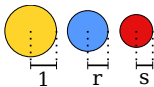


Packing P :
(in \mathbb{R}^2)

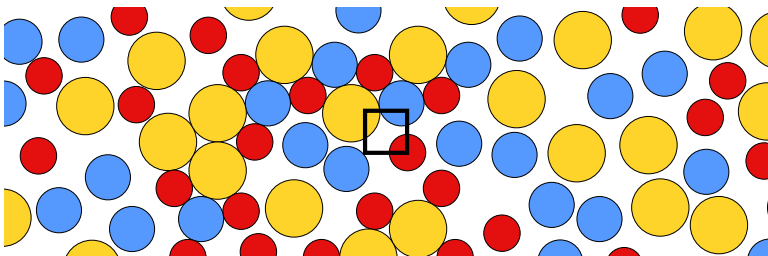


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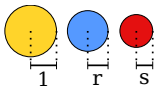


Density:

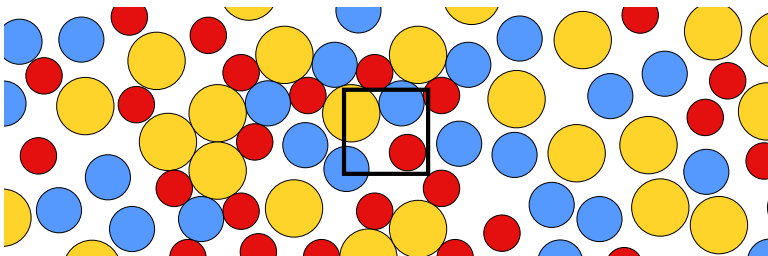
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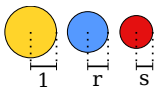


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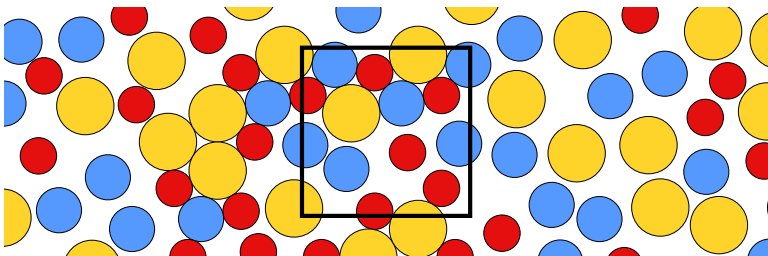
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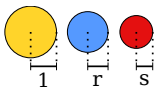


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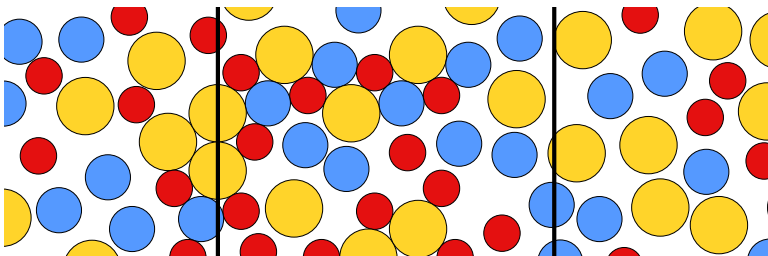
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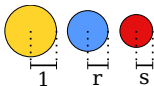


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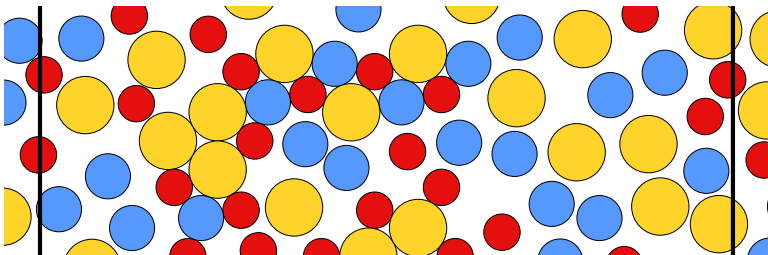
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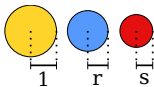


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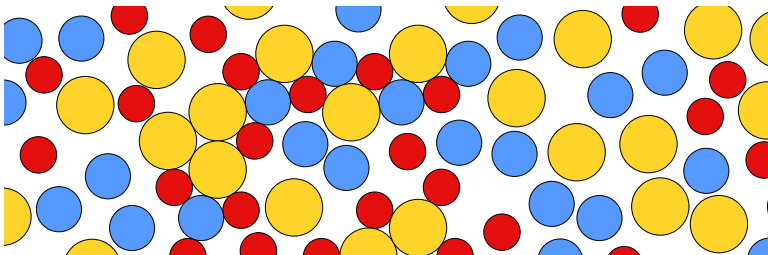
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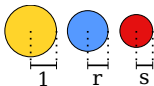


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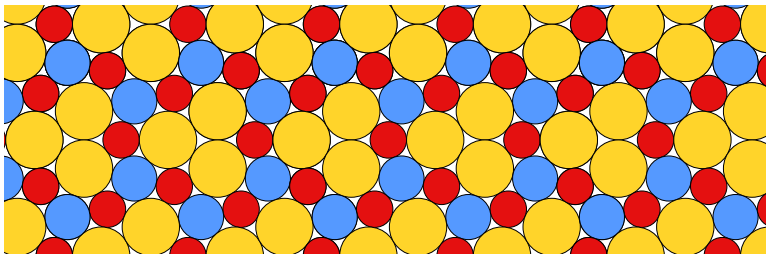
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
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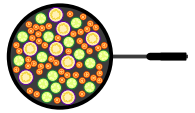
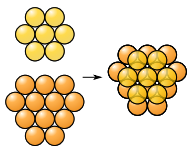
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Main Question

Given a finite set of discs (e.g.,   ),
which packing maximizes the density?

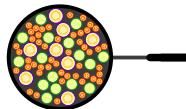
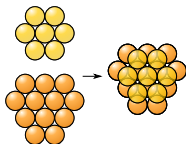
Motivation

- Packing fruits and vegetables

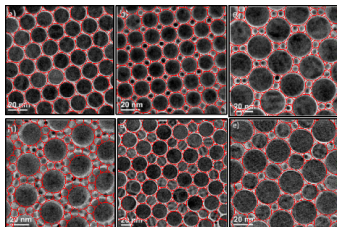
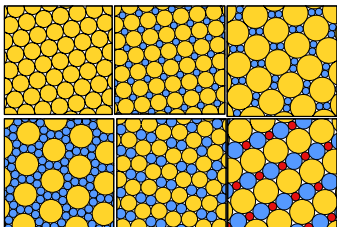
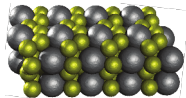


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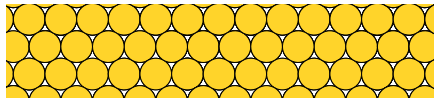


- Making compact materials



Binary and ternary superlattices self-assembled from colloidal nanodisks and nanorods.
Journal of the American Chemical Society, 137(20):6662–6669, 2015.

2D hexagonal ● -packing:



$$\delta = \frac{\pi}{2\sqrt{3}}$$

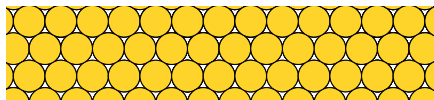
Lagrange, 1772

Hexagonal packing maximizes the density among lattice ● -packings.

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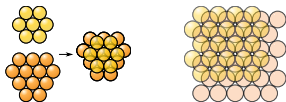
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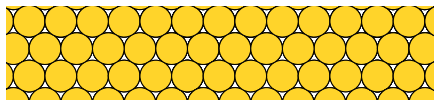
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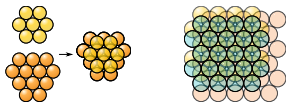
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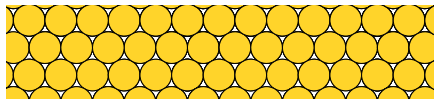
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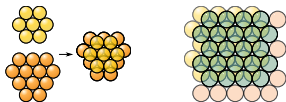
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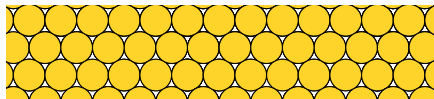
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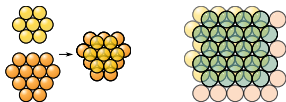
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Gauss, 1831

Hexagonal compact packings maximize the density among lattice ● -packings.

Hales, Ferguson, 1998–2014

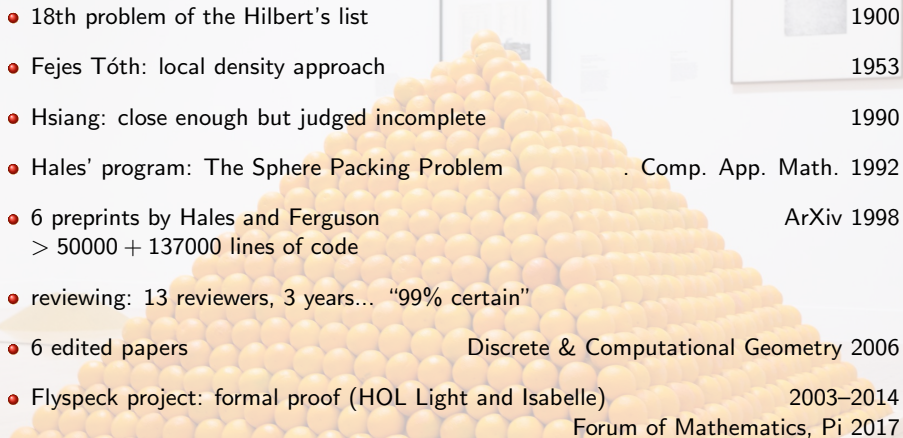
(Conjectured by Kepler, 1611)

Hexagonal compact packings maximize the density.

The proof of the Kepler Conjecture

Mathematics as we practice it is much more formally complete and precise than other sciences, but it is much less formally complete and precise for its content than computer programs

W. P Thurston 1994

- 
- 18th problem of the Hilbert's list
 - Fejes Tóth: local density approach 1900
 - Hsiang: close enough but judged incomplete 1953
 - Hales' program: The Sphere Packing Problem . Comp. App. Math. 1992
 - 6 preprints by Hales and Ferguson > 50000 + 137000 lines of code ArXiv 1998
 - reviewing: 13 reviewers, 3 years... "99% certain"
 - 6 edited papers Discrete & Computational Geometry 2006
 - Flyspeck project: formal proof (HOL Light and Isabelle) 2003–2014
Forum of Mathematics, Pi 2017

A packing is called **triangulated** if each “hole” is bounded by three tangent discs:



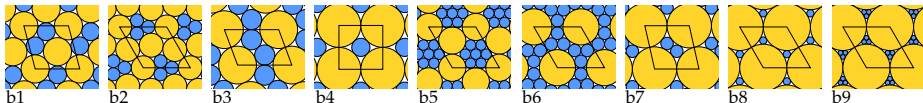
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●● Kennedy, 2006

(Packings by discs of radii $1, r$) There are 9 values of r allowing triangulated packings.



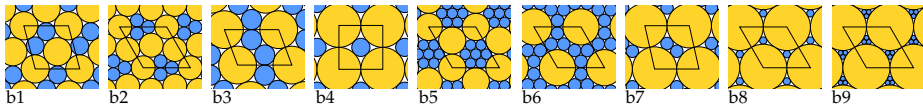
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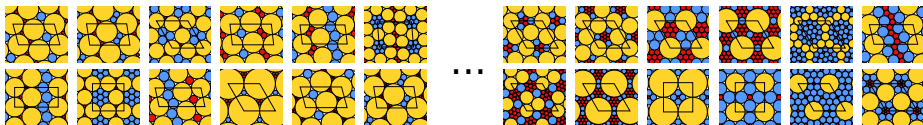
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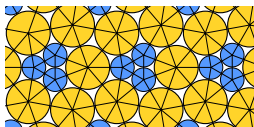
○●● Fernique, Hashemi, Sizova 2019

(Packings by discs of radii $1, r, s$) There are 164 pairs (r, s) allowing triangulated packings.

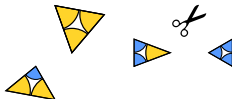


Even more motivation

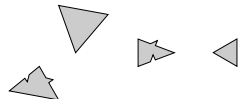
triangulated packings



~



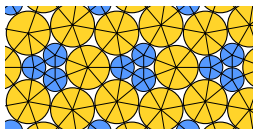
tilings by triangles
with local rules



density = weighted proportion of tiles

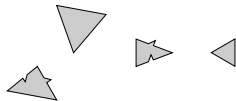
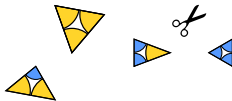
Even more motivation

triangulated packings



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tilings by triangles
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density = weighted proportion of tiles

Triangulated Packing Problem

algebraic numbers represented by polynomials and intervals

excludes hexagonal packing

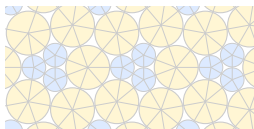
Given k disc radii $\overbrace{r_1, \dots, r_k}$, is there a triangulated packing of density $> \frac{\pi}{2\sqrt{3}}$

$\forall r_1, \dots, r_k$ with triangulated packings, one is periodic \Rightarrow **decidable**
(Wang algorithm: search for a period)

$\exists r_1, \dots, r_k$ whose triangulated packings are all aperiodic \Rightarrow **undecidable?**

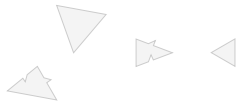
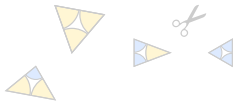
Even more motivation

triangulated packings



~

tilings by triangles
with local rules



density = weighted proportion of tiles

Dense Packing Problem

algebraic numbers represented by polynomials and intervals

Given k disc radii $\overbrace{r_1, \dots, r_k}$, is there a

excludes hexagonal packing

packing of density $> \frac{\pi}{2\sqrt{3}}$

$\forall r_1, \dots, r_k$ with dense packings, one is periodic
(interval arithmetic and subdivision until needed precision)

\Rightarrow

decidable

$\exists r_1, \dots, r_k$ whose dense packings are all aperiodic

\Rightarrow

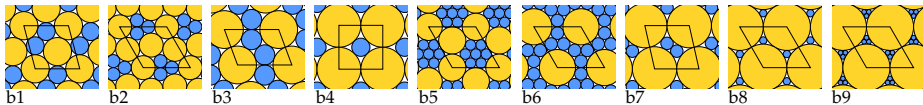
not possible!

Thue, 1910 (Toth, 1940)

The hexagonal packing maximizes the density.



Heppes 2000,2003; Kennedy 2004; Bedaride, Fernique, 2019



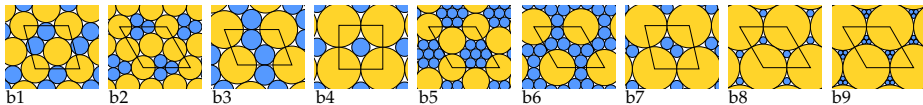
All these 9 triangulated packings maximize the density.

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Conjecture (Connelly, 2018)

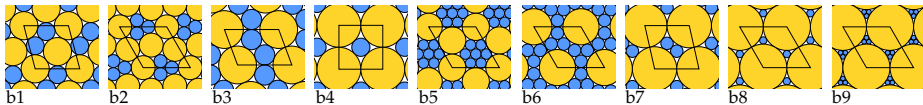
If a finite set of discs allows a **saturated** triangulated packing then the density is maximized on a saturated triangulated packing.

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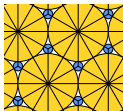
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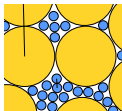
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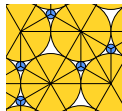
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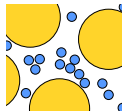
triangulated
saturated



non triangulated
saturated



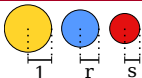
triangulated
non saturated



non triangulated
non saturated

Triangulated packings

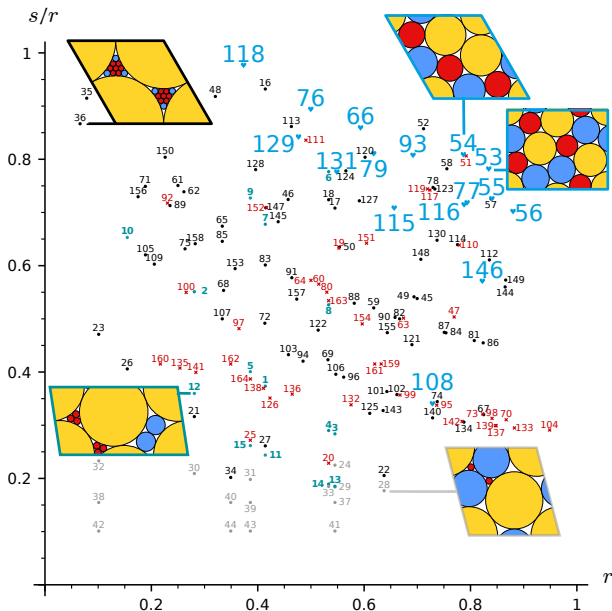
3 discs:



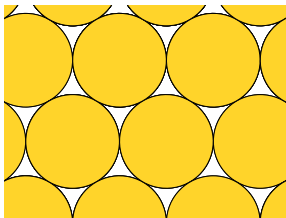
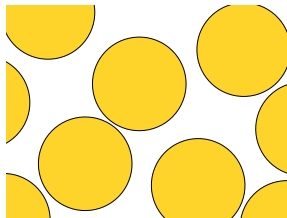
164 (r, s) allowing triangulated packings:

(Fernique, Hashemi, Sizova 2019)

- 15 cases: non saturated
- 40 cases: a binary **non** triangulated packing is denser (Fernique, P 2022)
- case 53: a ternary triangulated packing is densest (Fernique 2019)
- 15+15 more cases: a **ternary** or **binary** triangulated packing is densest (Fernique, P 2022)



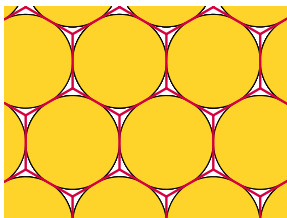
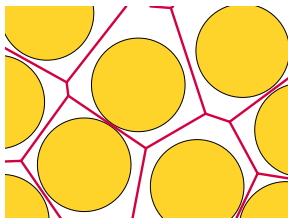
Delaunay triangulation

 P^* of density δ^*  P of density δ

Show that for any saturated packing P ,

$$\delta^* \geq \delta$$

Delaunay triangulation

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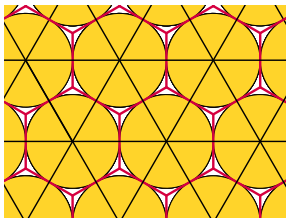
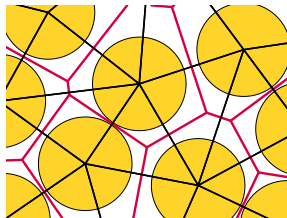
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Voronoi cell of a disc in a packing: set of points closer to this disc than to any other

Voronoi diagram of a packing: partition of the plane into Voronoi cells

Delaunay triangulation

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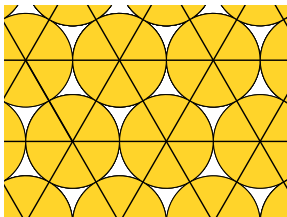
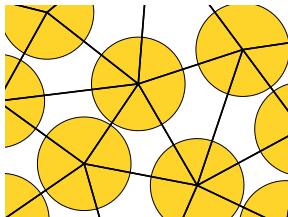
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Voronoi cell of a disc in a packing: set of points closer to this disc than to any other

Voronoi diagram of a packing: partition of the plane into Voronoi cells

Delaunay triangulation of a packing: dual graph of the Voronoi diagram

Delaunay triangulation

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Show that for any saturated packing P ,

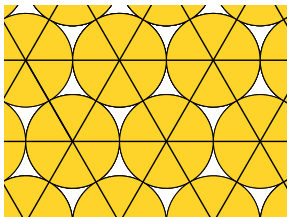
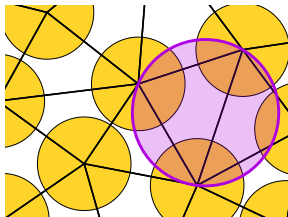
$$\delta^* \geq \delta$$

Voronoi cell of a disc in a packing: set of points closer to this disc than to any other

Voronoi diagram of a packing: partition of the plane into Voronoi cells

Delaunay triangulation of a packing: dual graph of the Voronoi diagram

Delaunay triangulation

 P^* of density δ^*  P of density δ

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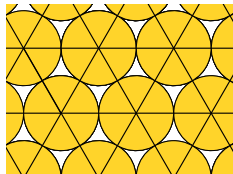
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Delaunay triangulation of a packing: dual graph of the Voronoi diagram

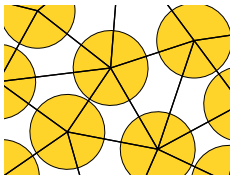
Main property: no points inside the **circumscribed circle** of any triangle

Density of a triangle Δ in a packing = its proportion covered by discs

$$\delta_{\Delta} = \frac{\text{area}(\Delta \cap P)}{\text{area}(\Delta)}$$



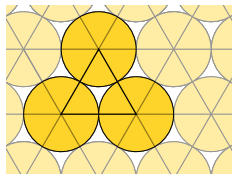
$$\delta^* = \delta_{\Delta^*} = \frac{\pi}{2\sqrt{3}}$$



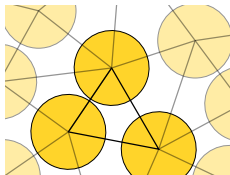
$$\forall \Delta, \delta_{\Delta} \leq \delta_{\Delta^*} = \delta^*$$

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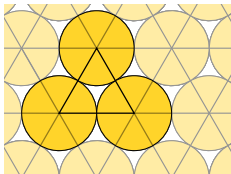
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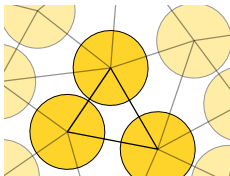
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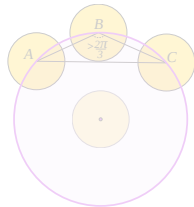
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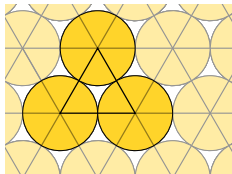


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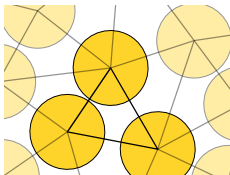


- The largest angle of any Δ is between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ $\hat{A} < \frac{\pi}{6} \Rightarrow R = \frac{|BC|}{2 \sin \hat{A}} \geq \frac{1}{\sin \hat{A}} > 2$
- The density of a triangle Δ : $\delta_{\Delta} = \frac{\pi/2}{\text{area}(\Delta)}$
- The area of a triangle ABC with the largest angle \hat{B} is $\frac{1}{2}|AB| \cdot |BC| \cdot \sin \hat{B}$ which is at least $\frac{1}{2} \cdot 2 \cdot 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$
- Thus the density of ABC is less or equal to $\frac{\pi/2}{\sqrt{3}} = \delta_{\Delta^*}$

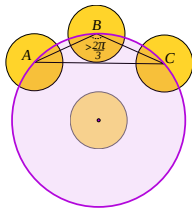
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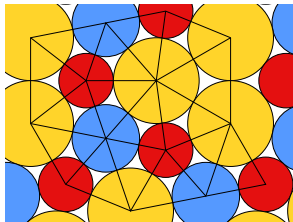
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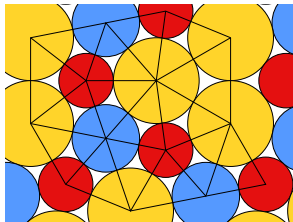
FM-triangulation of packing P^*

Triangles in P^* have different densities:

$$\delta\left(\begin{array}{c} \text{●} \\ \text{●} \end{array} \triangle \begin{array}{c} \text{●} \\ \text{●} \end{array}\right) \neq \delta\left(\begin{array}{c} \text{●} \\ \text{●} \end{array} \triangle \begin{array}{c} \text{●} \\ \text{●} \end{array}\right)$$

Hopeless to bound the density in each triangle...

What to do?



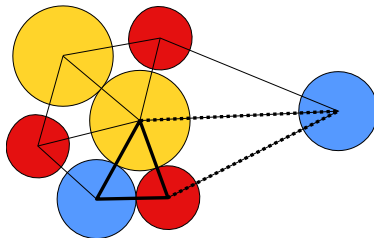
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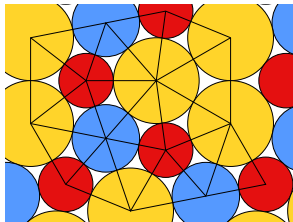
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Hopeless to bound the density in each triangle...
What to do?

Density redistribution:





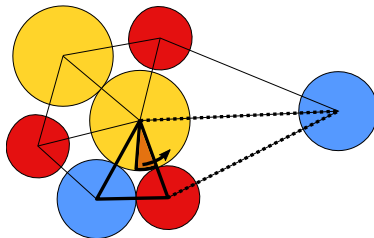
FM-triangulation of packing P^*

Triangles in P^* have different densities:

$$\delta\left(\begin{array}{c} \text{yellow} \\ \text{yellow} \end{array}\right) \neq \delta\left(\begin{array}{c} \text{blue} \\ \text{yellow} \end{array}\right)$$

Hopeless to bound the density in each triangle...
What to do?

Density redistribution:



Dense triangles “share their density” with neighbors

Emptiness instead of density



saturated triangulated packing P^*
density δ^* , FM-triangulation \mathcal{T}^*



saturated packing P with the same discs
density δ , FM-triangulation \mathcal{T}

Emptiness instead of density



saturated triangulated packing P^*
density δ^* , FM-triangulation \mathcal{T}^*



saturated packing P with the same discs
density δ , FM-triangulation \mathcal{T}

Density function is not additive: $\delta \left(\begin{array}{c} \text{blue} \\ \text{red} \\ \text{yellow} \end{array} \right) + \delta \left(\begin{array}{c} \text{red} \\ \text{red} \\ \text{yellow} \end{array} \right) \neq \delta \left(\begin{array}{c} \text{blue} \\ \text{red} \\ \text{red} \\ \text{yellow} \end{array} \right) \rightarrow$

Emptiness instead of density



saturated triangulated packing P^*
density δ^* , FM-triangulation \mathcal{T}^*



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Emptiness of a triangle $\Delta \in \mathcal{T}$: $E(\Delta) = \delta^* \times \text{area}(\Delta) - \text{area}(\Delta \cap P)$

$E(\Delta) > 0$ iff the density of Δ is less than δ^*

$E(\Delta) < 0$ iff the density of Δ is greater than δ^*

Additive!

Emptiness instead of density



saturated triangulated packing P^*
density δ^* , FM-triangulation \mathcal{T}^*



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Additive!

$$\delta^* \geq \delta \Leftrightarrow \sum_{\Delta \in \mathcal{T}} E(\Delta) \geq 0$$

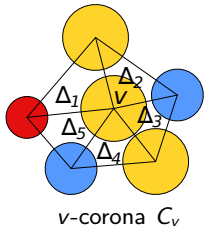
We construct a **potential** $U(\Delta) := \overbrace{\dot{U}_\Delta^A + \dot{U}_\Delta^B + \dot{U}_\Delta^C}^{\text{vertices}}$ such that

\forall triangle $\Delta \in \mathcal{T}$, $U(\Delta) \leq E(\Delta)$ (Δ)

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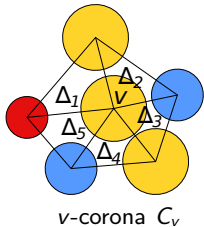
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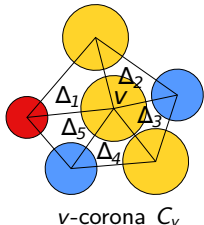


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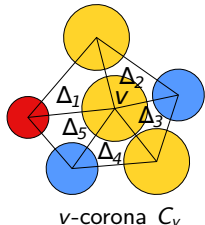
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If such U exists then $\delta^* \geq \delta$

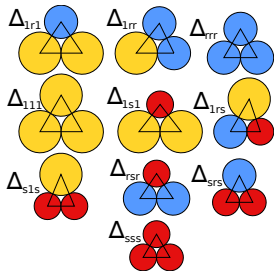
How to construct it and prove (Δ) , (\bullet) ?

Choosing U to assure (●)

(inspired by Kennedy 2005)

 Δ_{xyz}
 \widehat{xyz}
 V_{xyz}

tight triangle: tangent discs of radii x, y, z
 angle of Δ_{xyz} in the center of the y -disc
 potential of Δ_{xyz} in the center of the y -disc

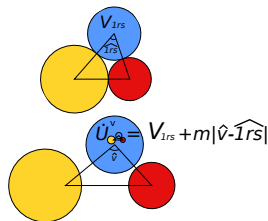


Choosing U to assure (●)

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 Δ_{xyz} tight triangle: tangent discs of radii x, y, z \widehat{xyz} angle of Δ_{xyz} in the center of the y -disc V_{xyz} potential of Δ_{xyz} in the center of the y -discpotential of a triangle Δ in v :

$$\dot{U}_{\Delta}^v := V_{xyz} + m|\hat{v} - \widehat{xyz}|$$

measures how "far" Δ is from being tight

Choosing U to assure (●)

(inspired by Kennedy 2005)

$$\Delta_{xyz}$$

$$\widehat{xyz}$$

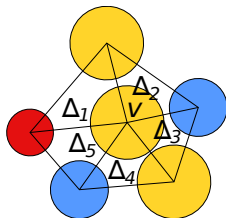
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Choose m to satisfy $\sum_{\Delta \in C_v} \dot{U}_{\Delta}^v \geq \sum_{\substack{x,y,z \\ \text{disc radii of} \\ \Delta \in C_v}} V_{xyz} + m \times |2\pi - \sum_{\substack{x,y,z \\ \text{disc radii of} \\ \Delta \in C_v}} \widehat{xyz}| \geq 0$ for all coronas C_v

Choosing U to assure (●)

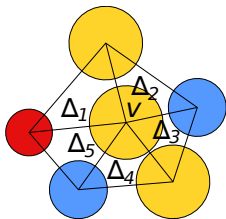
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angle values do not matter \Rightarrow

sequence of disc radii $S(C_v)$

FM-triangulation \Rightarrow

bounded $|S(C_v)|$

finite number of linear inequalities on m
 \Rightarrow computer search

(Δ)

Defining U , we make it as small as possible keeping it positive around any vertex (●)

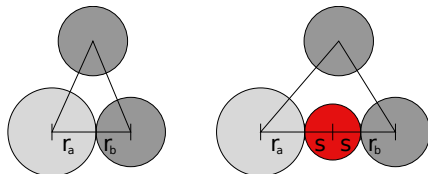
(Δ) : How to check $E(\Delta) \geq U(\Delta)$ on each triangle Δ ? (there is a continuum of them)

(Δ)

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FM-triangulation properties + saturation \Rightarrow uniform bound on edge length



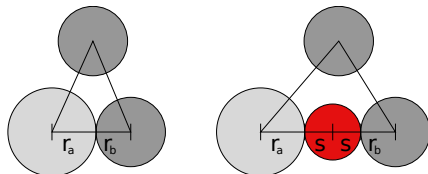
$$r_a + r_b \leq c \leq r_a + r_b + 2s$$

(Δ)

Defining U , we make it as small as possible keeping it positive around any vertex (●)

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FM-triangulation properties + saturation \Rightarrow uniform bound on edge length



$$r_a + r_b \leq c \leq r_a + r_b + 2s$$

interval arithmetic

verify $E(\Delta_{a,b,c}) \geq U(\Delta_{a,b,c})$ where

$$a = [r_c + r_b, r_c + r_b + 2s] \quad b = [r_a + r_c, r_a + r_c + 2s] \quad c = [r_a + r_b, r_a + r_b + 2s]$$

not precise enough (intervals intersect) \rightarrow dichotomy

Inequalities and interval arithmetic

A representation of a number x is an interval I whose endpoints are exact values representable in a computer memory and such that $x \in I$.

```
sage: x = RIF(0,1) # Interval [0,1]
sage: x<2 #  $\forall t \in [0,1], t < 2$ 
True
sage: (x+x).endpoints() # [0,1]+[0,1]
(0.0, 2.0)

sage: Ipi = 4*arctan(RIF(1)) # Interval for  $\pi$ 
(3.14159265358979, 3.14159265358980)
sage: sin(Ipi).endpoints() # Interval for  $\sin(\pi)$ 
(-3.21624529935328e-16, 1.22464679914736e-16)
```

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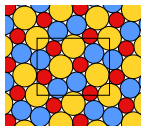
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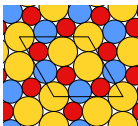
Two intersecting intervals are incomparable:

```
sage: sin(Ipi)<=x
False
sage: sin(Ipi)>=x # These intervals intersect
False
```

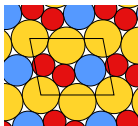
Our proof worked for these cases:



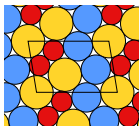
53



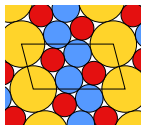
54



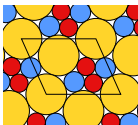
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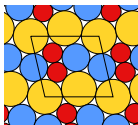
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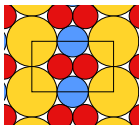
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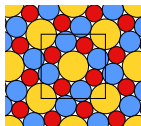
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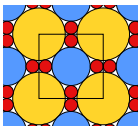
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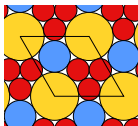
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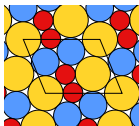
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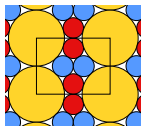
108



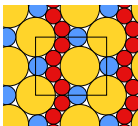
115



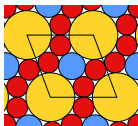
116



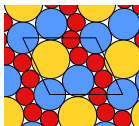
118



129

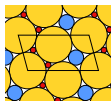


131

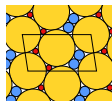


146

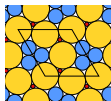
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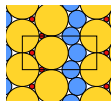
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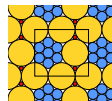
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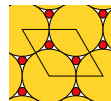
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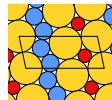
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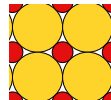
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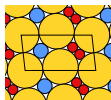
b_8



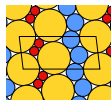
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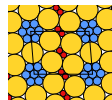
b_4



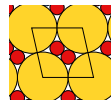
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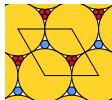
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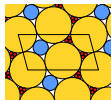
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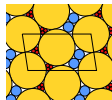
b_7



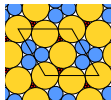
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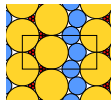
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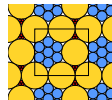
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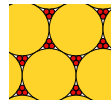
13



14



15



b_9

Steps of the proof

δ^* denotes the maximal density

●|●●|●●● 1 partition space into “small” cells



















FM-triangulation

●|●●|●●● 2 find a suitable function to represent the density

emptiness E






















Steps of the proof

 δ^* denotes the maximal density

- | | | |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------|
|     1 | partition space into “small” cells | FM-triangulation |
|     2 | find a suitable function to represent the density | emptiness E |
|    3 | distribute the density among the vertices in each cell | potential \dot{U} |
|    4 | verify that the redistributed density $\leq \delta^*$
check all possible local configurations |  choice of m in \dot{U}
run through all coronas |
|    5 | treat special cases which can not be treated by this technique
which are usually configurations close to local maxima | ϵ -triangles
coronas of \mathcal{T}^* |
-

Steps of the proof

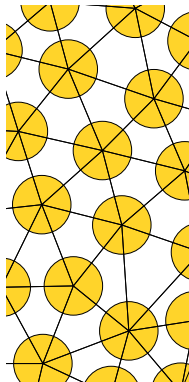
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- | | | |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------|
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-
- | | | |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------|---------------------|
|    6 | verify that the sum of vertex densities in a cell \geq its density
check all possible cells | interval arithmetic |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------|---------------------|

Space partition

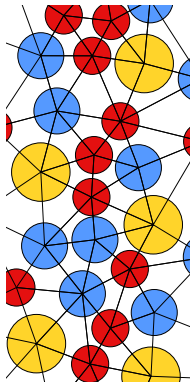
2D, ●

Delaunay
triangulation



2D, ●●●

FM-triangulation \mathcal{T}
(weighted Delaunay triangulation)



3D, ●●●

Voronoi cells + Delaunay simplices

HF-partition \mathcal{P}

consists of several types of simplices and
modified Voronoi cells

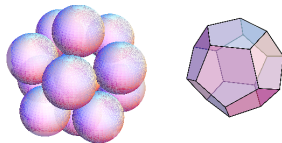
(Sphere packings II. A formulation of the Kepler Conjecture)

only Voronoi cells or
only Delaunay simplices



local configurations denser than δ^*

regular dodecahedron for Voronoi cells
(as in dodecahedral conjecture)

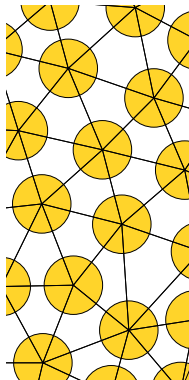


Hales and McLaughlin 1998

Space partition

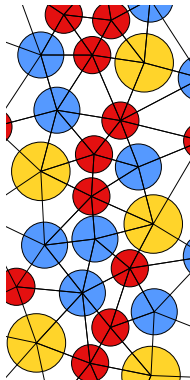
2D, ●

Delaunay
triangulation



2D, ●●●

FM-triangulation \mathcal{T}
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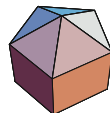
(Sphere packings II. A formulation of the Kepler Conjecture)

only Voronoi cells or
only Delaunay simplices

↓

local configurations denser than δ^*

pentagonal prism for Delaunay simplices



Ferguson 2006

Suitable function to measure the density

2D, ●

2D, ●●●

3D, ●●

density
 δ

emptiness

compression

$$E(\Delta) := \delta^* \times \text{area}(\Delta) - \text{area}(\Delta \cap P)$$

$$\Gamma(R) := \text{vol}(R \cap P) - \delta_{\text{oct}} \times \text{vol}(R)$$

$$\delta_{\text{oct}} := \delta(\text{tight regular octahedron}) < \delta^*$$

$$\delta^* = \frac{\delta_{\text{tet}}}{3} + 2\frac{\delta_{\text{oct}}}{3}$$

Suitable function to measure the density

2D, ●

2D, ●●●

3D, ●●

density
 δ

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$$\delta_{\text{oct}} := \delta(\text{tight regular octahedron}) < \delta^*$$

$$\delta^* = \frac{\delta_{\text{tet}}}{3} + 2 \frac{\delta_{\text{oct}}}{3}$$

An additive function reflecting the density:

 $\Gamma(R) < 0$ iff the density of R is less than δ_{oct} $\Gamma(R) > 0$ iff the density of R is greater than δ_{oct} $\delta \leq \delta^* \iff \Gamma$ is “low enough” on each small region

Redistribution of emptiness and compression

2D, ●●●

vertex potential

for $\Delta \in \mathcal{T}, v \in \Delta,$

$$\dot{U}_{\Delta}^v = V_{xyz} + m \times |\hat{v} - \hat{v}_{xyz}|$$

where x, y, z are disc radii of Δ \hat{v} is the angle of Δ in v and $V_{xyz}, m, \hat{v}_{xyz}$ are constants

3D, ●

score

for $R \in \mathcal{P}, v \in R,$ $\sigma(R, v)$ depends on the type of R if R is a Voronoi cell of v ($R = \text{Vor}(v)$),then $\sigma(R, v) = 4\Gamma(R)$ and $\sigma(R, w) = 0$ for $w \neq v$ if R is a simplex, σ varies in function of its properties and depends on Γ

Redistribution of emptiness and compression

2D, ●●●●

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$$U(\Delta) = U_{\Delta}^A + U_{\Delta}^B + U_{\Delta}^C \leq E(\Delta)$$

i.e. U is easy to manipulate and is at most E (lower approximation)

3D, ●●●●

score

for $R \in \mathcal{P}, v \in R,$ $\sigma(R, v)$ depends on the type of R if R is a Voronoi cell of v ($R = \text{Vor}(v)$),then $\sigma(R, v) = 4\Gamma(R)$ and $\sigma(R, w) = 0$ for $w \neq v$

if R is a simplex, σ varies in function of its properties and depends on Γ

$$\sum_{v \in R} \sigma(R, v) = 4\Gamma(R)$$

i.e. score of a region always equals to $4 \times$ compression

(Sphere packings II. A formulation of the Kepler Conjecture)

Verify that redistribution $\leq \delta^*$ around each vertex

2D, ●●●

 for each $v \in \mathcal{T}$, $\sum_{\Delta \in C_v} \dot{U}_\Delta^v \geq 0$

 configuration around a vertex –
corona C

 combinatorial representation of C –
sequence of disc radii $S(C)$

 choose m to satisfy

$$\sum_{\substack{x,y,z \\ \text{disc radii of} \\ \Delta \in C}} V_{xyz} + m \times \left| 2\pi - \sum_{\substack{x,y,z \\ \text{disc radii of} \\ \Delta \in C}} \widehat{xyz} \right| \geq 0$$

 for all coronas C

 FM-triangulation \Rightarrow bounded $|S(C)|$
 \Rightarrow finite number of inequalities on m
 \Rightarrow computer search

(linear programming)

3D, ●●

 for each $v \in \mathcal{P}$, $\sum_{R \in D_v} \sigma(R, v) \leq 8pt$

 configuration around a vertex –
decomposition star D

 combinatorial representation of D –
graph $G(D)$

 conditions on geometry of D “easily” implying
 $\sigma(D) \leq 8pt$ (interval arithmetics)

(Sphere packings IV. Detailed bounds)
tame graphs – 25 000 graphs of the remaining
 D have restricted geometry $\Rightarrow \max \sigma(D) \leq 8pt$
 (linear programming)

(Sphere packings VI. Tame graphs and linear programs)

 except **pentagonal prism graph**,

(Sphere packing V. Pentahedral prisms)
FCC graph and **HCP graph**
(Sphere packings III. Extremal cases)

Verify that redistribution $\leq \delta^*$ around each vertex

2D, ●●●●

for each $v \in \mathcal{T}$, $\sum_{\Delta \in \mathcal{C}_v} \dot{U}_\Delta^v \geq 0$ configuration around a vertex –
corona C combinatorial representation of C –
sequence of disc radii $S(C)$ choose m to satisfy

$$\sum_{\substack{x,y,z \\ \text{disc radii of} \\ \Delta \in C}} V_{xyz} + m \times |2\pi - \sum_{\substack{x,y,z \\ \text{disc radii of} \\ \Delta \in C}} \widehat{xyz}| \geq 0$$

for all coronas C FM-triangulation \Rightarrow bounded $|S(C)|$
 \Rightarrow finite number of inequalities on m
 \Rightarrow computer search

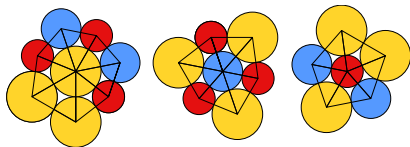
(linear programming)

3D, ●●

for each $v \in \mathcal{P}$, $\sum_{R \in \mathcal{D}_v} \sigma(R, v) \leq 8pt$ configuration around a vertex –
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(linear programming)*(Sphere packings VI. Tame graphs and linear programs)*except **pentagonal prism graph**,*(Sphere packing V. Pentahedral prisms)***FCC graph** and **HCP graph***(Sphere packings III. Extremal cases)*

Extremal cases

2D, ●●●

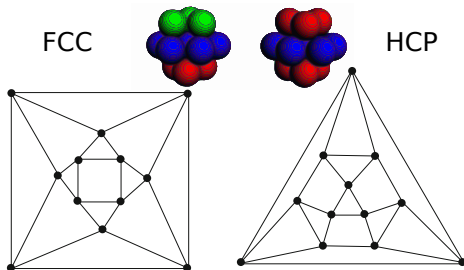


for the coronas of \mathcal{T}^* , $\sum_{x,y,z} V_{xyz} := 0$
 x,y,z
disc radii of
 $\Delta \in y$ -corona

for tight triangles, $U(\Delta_{xyz}) := E(\Delta_{xyz})$

ϵ -triangles – triangles close to tight
 \Rightarrow potential close to emptiness

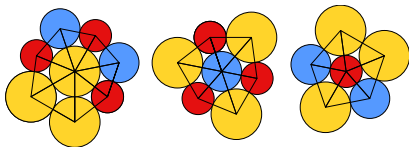
3D, ●●



FCC and HCP decomposition stars have maximal score
 \Rightarrow close configurations have high score

Extremal cases

2D, ●●●



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ϵ -triangles – triangles close to tight
 \Rightarrow potential close to emptiness

derivatives on side lengths x_i :

$$\min_{T_\epsilon} \frac{\partial E}{\partial x_i} \Delta x_i \geq \max_{T_\epsilon} \frac{\partial U}{\partial x_i} \Delta x_i$$

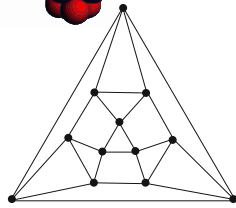
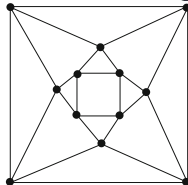
(interval arithmetic)

3D, ●●●●

FCC



HCP



FCC and HCP decomposition stars have maximal score

\Rightarrow close configurations have high score

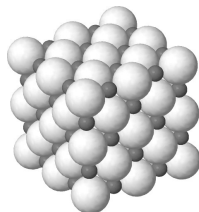
derivatives to prove that FCC and HCP are local maxima

(Sphere packings III. Extremal cases)

Locally halite packing

triangulated \rightarrow simplicial
(contact graph is a pure simplicial 3-complex)

halite packing is formed by a close-packing of unit spheres where all octahedral holes are filled with spheres of radius $\sqrt{2}-1$



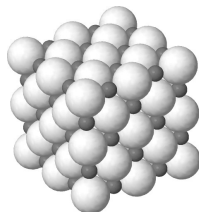
Fernique, 2019

The only simplicial packings by two sizes of spheres in 3D are halite packings.

Locally halite packing

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Fernique, 2019

The only simplicial packings by two sizes of spheres in 3D are halite packings.

Conjecture

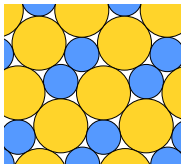
Halite packings maximize the density.

- simplicial partition
 - vertex or edge potential?
 - sphere triangulations or necklaces?
-
- bound density inside a tetrahedron (dimension reduction+derivatives)

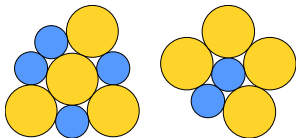
Thank you for your attention! :-)

How to find triangulated packings

A packing is triangulated



Each disc has a "corona"

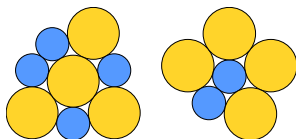
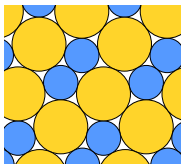


How to find triangulated packings

A packing is triangulated

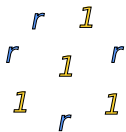


Each disc has a “corona”



To find disc sizes with triangulated packings, we run through all possible combinations of symbolic coronas of two discs (finite number)

Symbolic corona



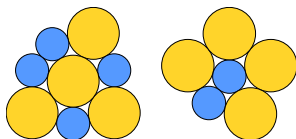
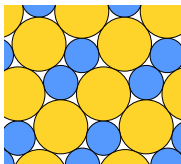
(Fernique, Hashemi, Sizova 2019)

How to find triangulated packings

A packing is triangulated

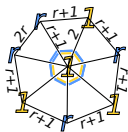
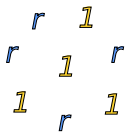


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Symbolic corona



Value of r

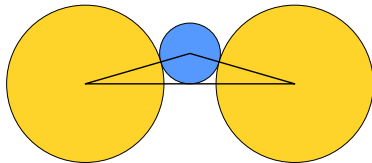
$$6 \times \widehat{11r} + 1 \times \widehat{r1r} = 2\pi$$

$$r \approx 0.63$$

(Fernique, Hashemi, Sizova 2019)

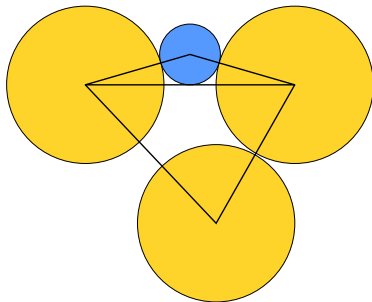
Edge Potentials

Stretched triangles feature low emptiness but high vertex potential:



Edge Potentials

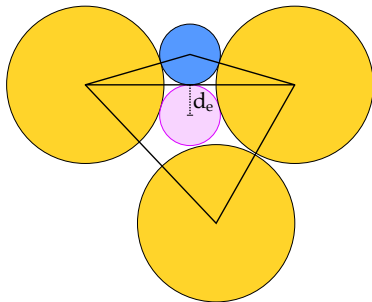
Stretched triangles feature low emptiness but high vertex potential:



Their neighbors feature high emptiness, so we can “balance” potential between them.

Edge Potentials

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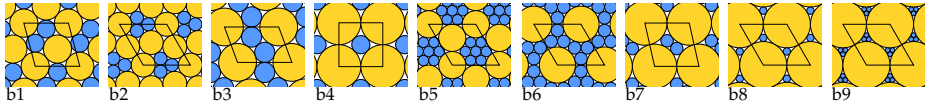
Their neighbors feature high emptiness, so we can “balance” potential between them.

To do this, we introduce edge potential: $U(\Delta) := \dot{U}_\Delta + \bar{U}_\Delta$

$$\bar{U}_\Delta = \bar{U}_\Delta^{e_1} + \bar{U}_\Delta^{e_2} + \bar{U}_\Delta^{e_3}$$

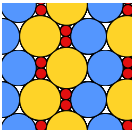
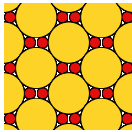
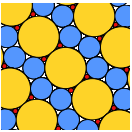
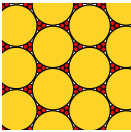
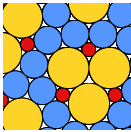
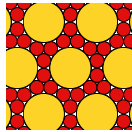
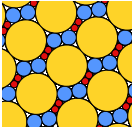
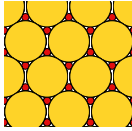
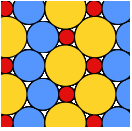
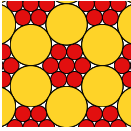
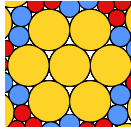
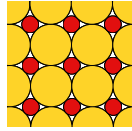
$$\bar{U}_\Delta^e := \begin{cases} qd_e & \text{if } |e| > l \\ 0 & \text{otherwise} \end{cases}$$

40 counter examples



When the ratio of two discs is close enough to the ratio in a dense binary packing, we can pack these disc in a similar manner (non triangulated) and still get high density

Examples:

	triangulated ternary packing	counter example using only 2 discs	triangulated ternary packing	counter example using only 2 discs	triangulated ternary packing	counter example using only 2 discs
70	 $\delta=0.921134$ $s=0.268266$	 $\delta=0.926300$ $s=0.268266$	 $\delta=0.931369$ $s=0.121445$	 $\delta=0.937371$ $s=0.121445$	 $\delta=0.914301$ $s=0.337336$	 $\delta=0.917953$ $s=0.337336$
97	 $\delta=0.931017$ $s=0.175341$	 $\delta=0.932390$ $s=0.175341$	 $\delta=0.915670$ $s=0.386662$	 $\delta=0.919703$ $s=0.386662$	 $\delta=0.914148$ $s=0.409604$	 $\delta=0.917188$ $s=0.409604$