

Some questions around high genus maps

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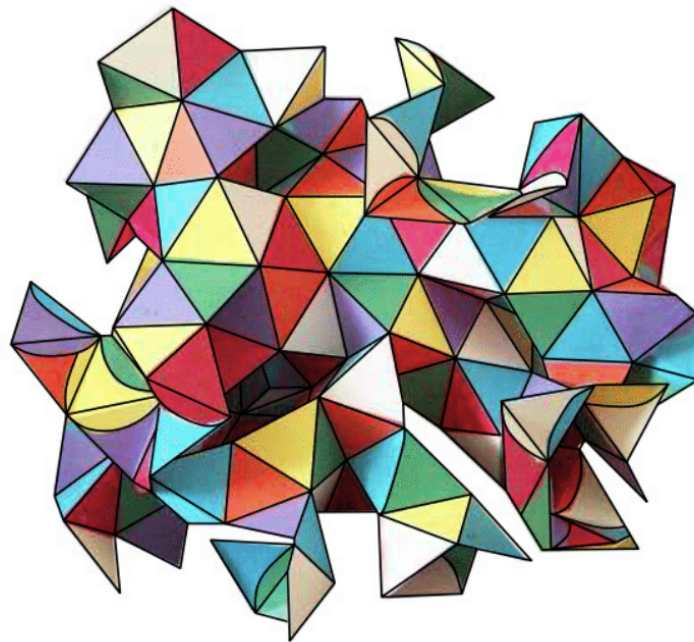
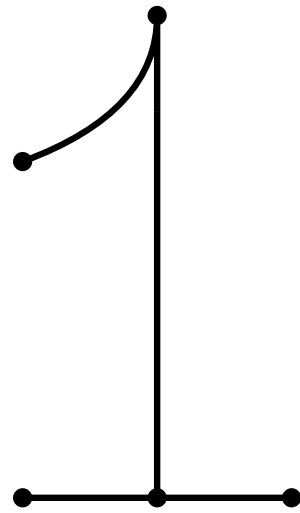


image : N. Curien



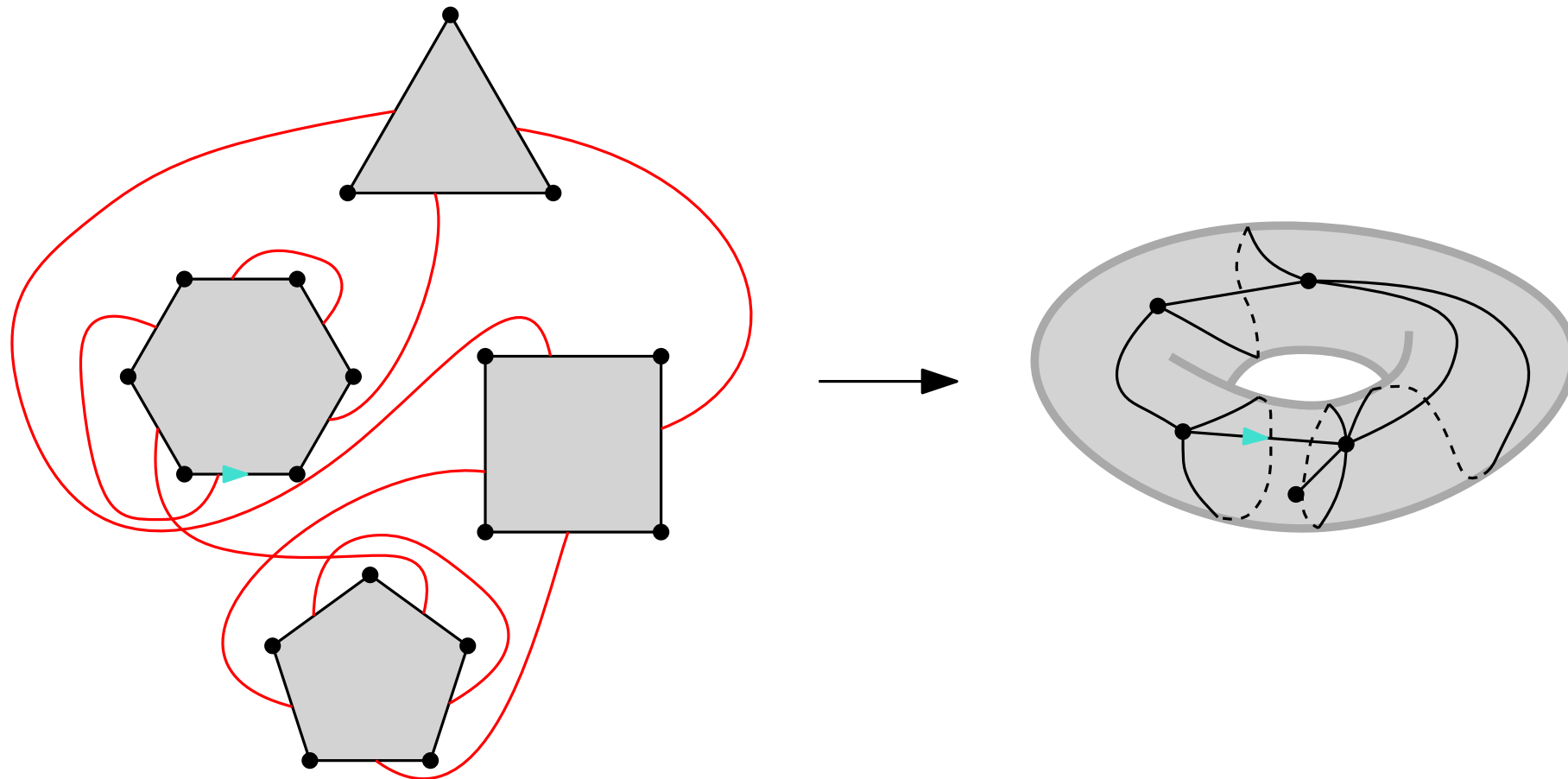
Maps: definitions and history

Definition : maps

Map = gluing of polygons along their edges to create a (compact, connected, oriented) surface

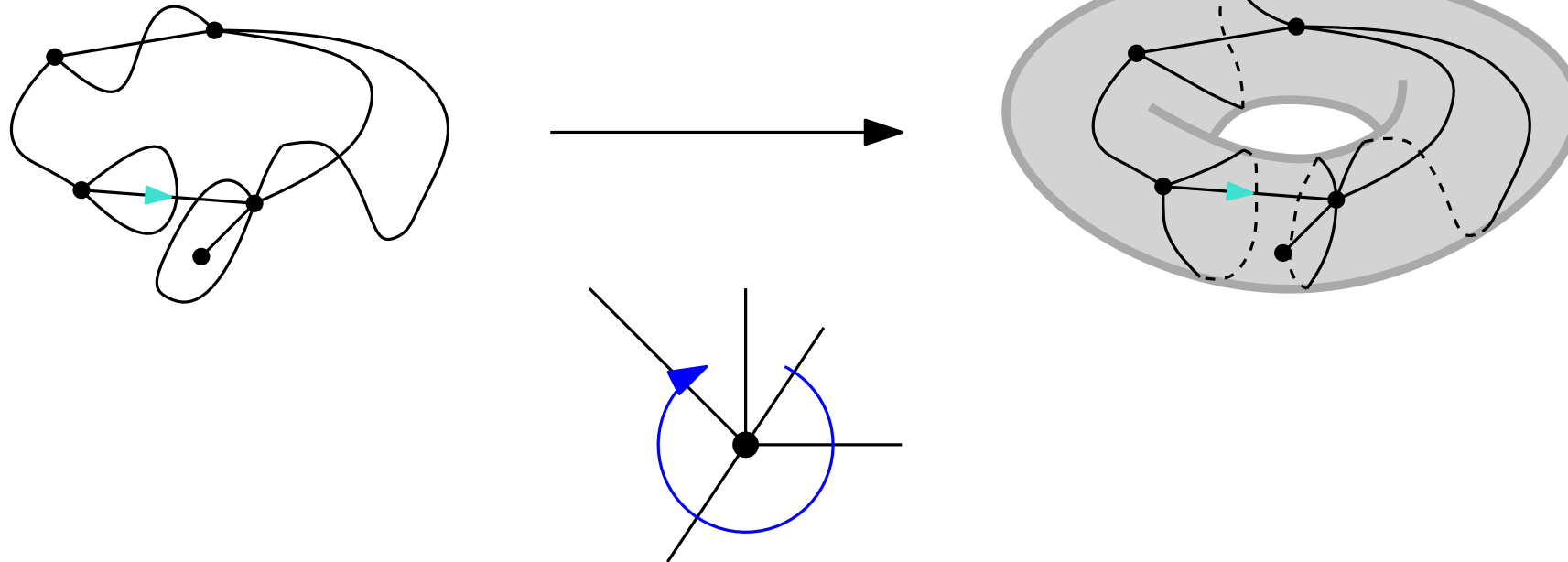
Genus g of the map = genus of the surface = # of handles

Rooted = an oriented edge is distinguished



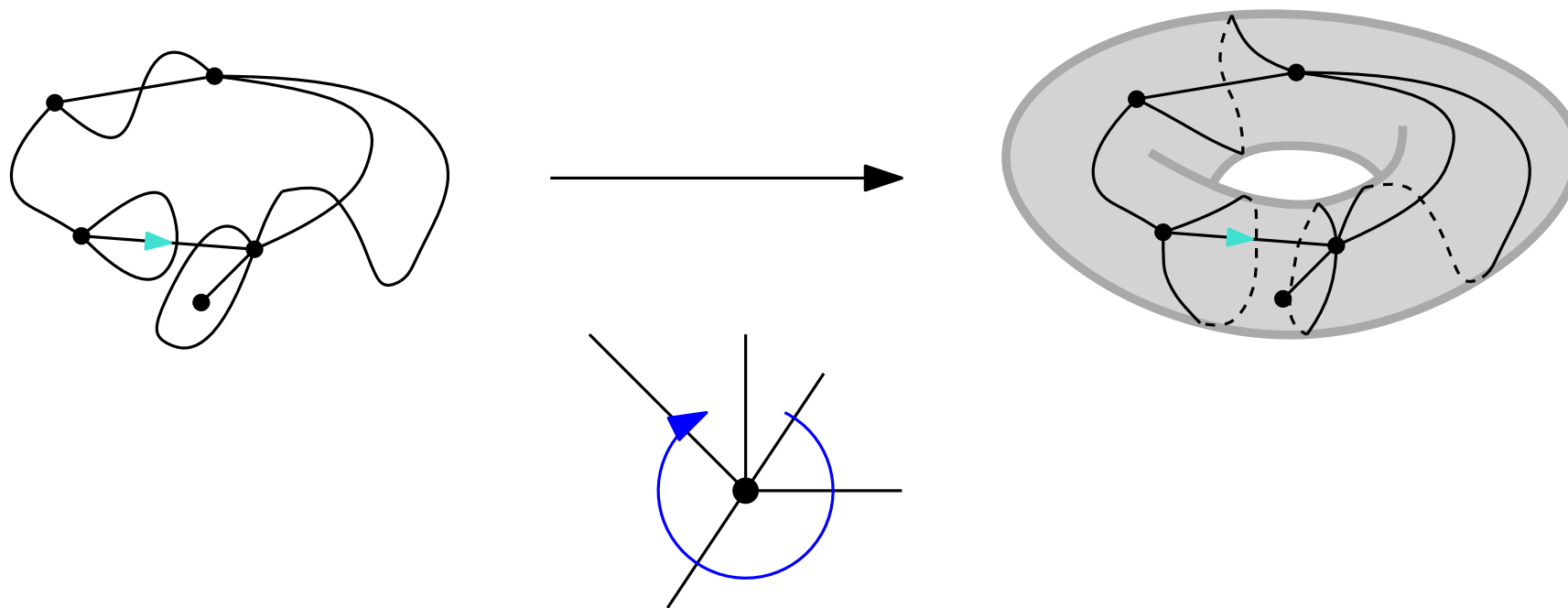
Definition : maps

Alternative definition = graphs embedded on surfaces



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(alternative definition = factorisations of permutations)

Large random maps

Maps = good model of discrete geometry
→ what does a large random map look like ?

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Family \mathcal{M}_n of random maps parametrized by size n .
Take $\mathbf{M}_n \in \mathcal{M}_n$ i.i.d., study its properties/limit as $n \rightarrow \infty$

Large random maps

Example: $\mathcal{M}_n =$ planar (genus 0) maps with n edges

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→ "scaling limit" = Brownian map [Le Gall, Miermont '13]

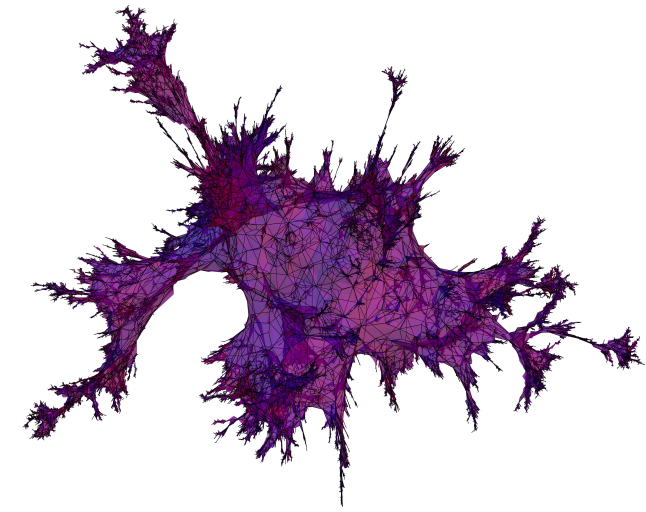


image : J. Bettinelli

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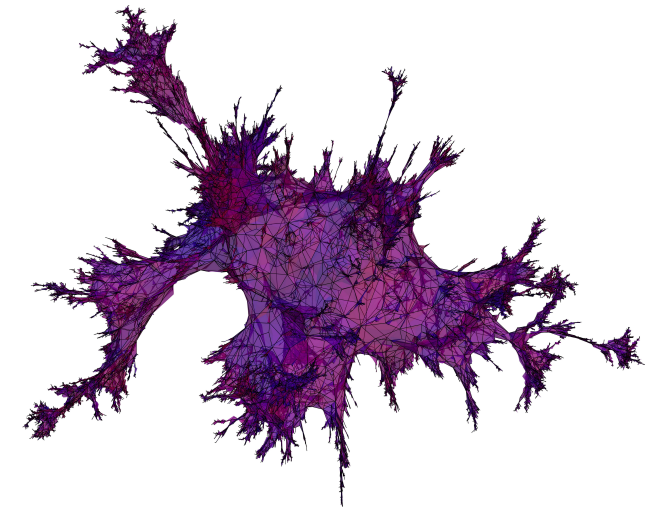
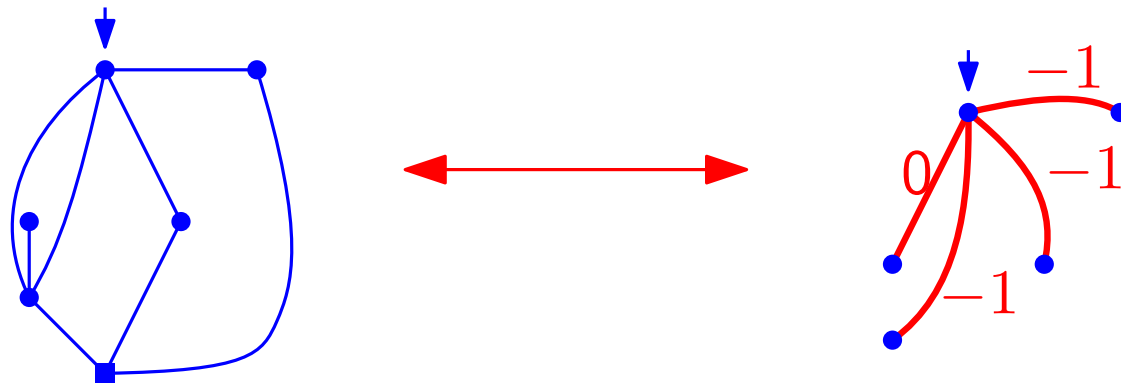


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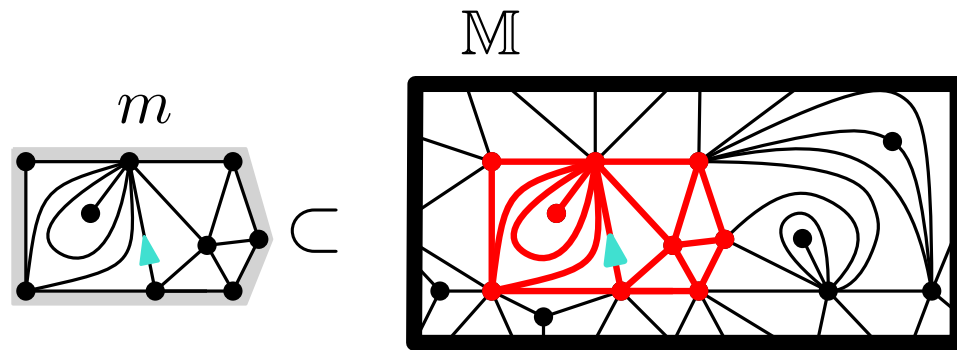
Fundamental tool : Schaeffer's bijection

Planar maps \leftrightarrow labeled trees



Local convergence

Local convergence = limit law of
finite neighborhoods of the root

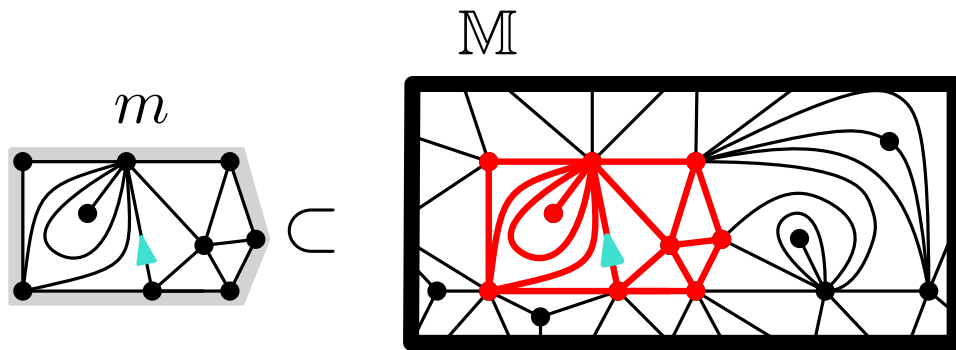


$\mathbf{M}_n \rightarrow M$ iff for all finite m :

$$\mathbb{P}(m \subset \mathbf{M}_n) \rightarrow \mathbb{P}(m \subset M)$$

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$\mathcal{M}_n \rightarrow \mathbb{M}$ iff for all finite m :
 $\mathbb{P}(m \subset \mathcal{M}_n) \rightarrow \mathbb{P}(m \subset \mathbb{M})$

$\mathcal{M}_n =$ triangulations (all faces of degree 3) of the sphere with $2n$ triangles

[Angel, Schramm '02] : the sequence \mathcal{M}_n converges to the Uniform Infinite Planar Triangulation (UIPT).

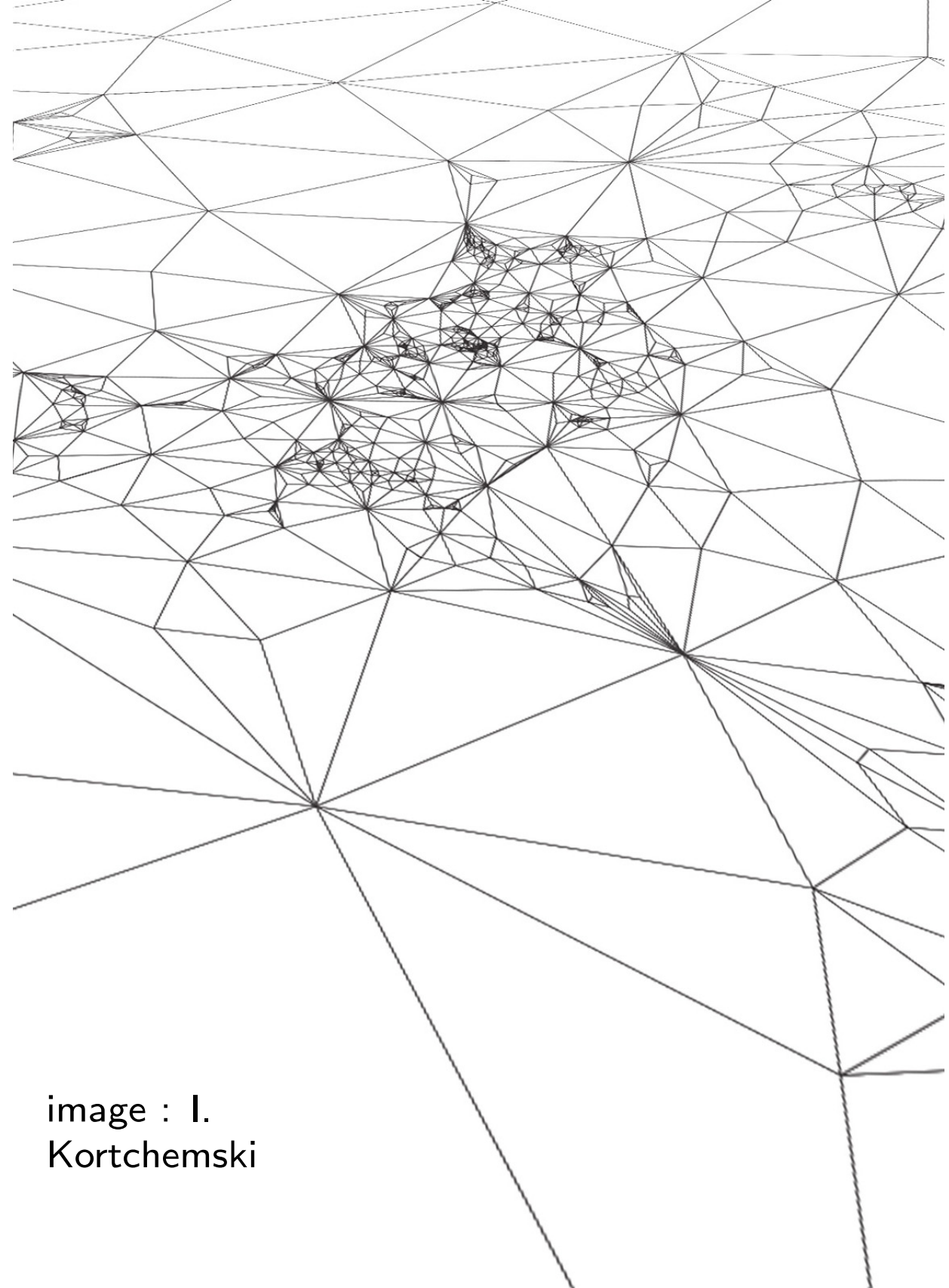
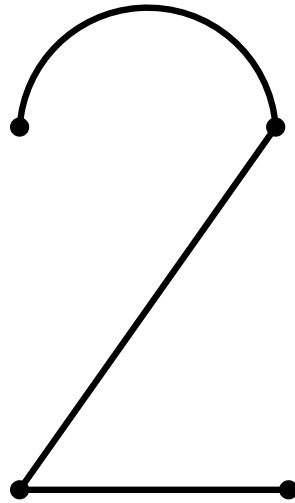


image : I.
Kortchemski



High genus maps

What is a high genus map ?

Take $(g_n)_{n \in \mathbb{N}}$ s.t. $\frac{g_n}{n} \rightarrow \theta \in (0, 1/2)$

For instance, $\mathcal{M}_n =$ triangulations of genus g_n with $2n$ triangles

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Genus goes to infinity very fast !

Average degree of a vertex = $\frac{6}{1-2\theta}$

→ related to average curvature

→ expected hyperbolic behaviour

“Toy model”: one faced maps

For $\mathcal{M}_n =$ maps with one face, genus g_n and n edges

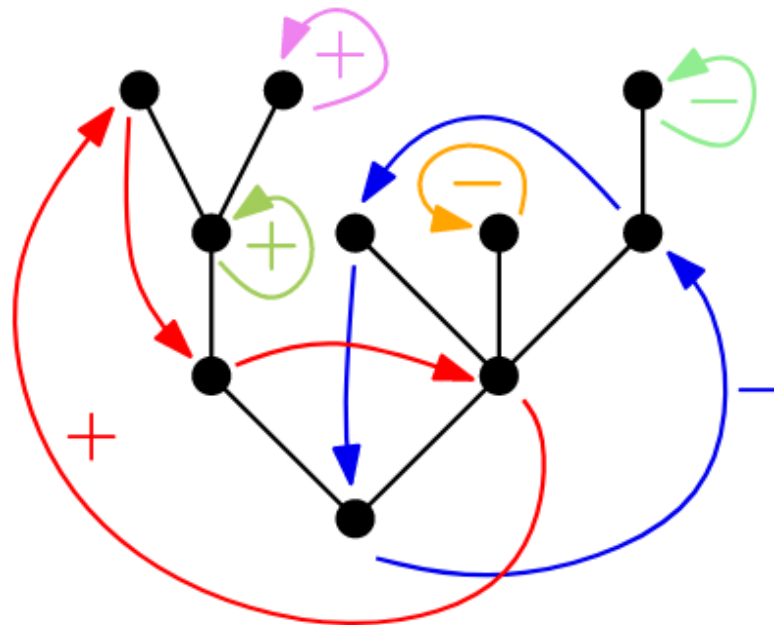
- local limit = “hyperbolic tree” (supercritical GW) [Angel–Chapuy–Curien–Ray 15]
- diameter = $\Theta(\log n)$ [Ray 15]
- short cycles follow a Poisson law [Janson–L. 21]
- large expander subgraphs [L. 21]

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Fundamental tool = bijection between one face maps and decorated trees
[Chapuy–Féray–Fusy 14]



The local limit of high genus triangulations

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Conjecture [Benjamini, Curien], Theorem

[Budzinski, L. 19] :

Local limit of $\mathcal{M}_n = \text{PSHT}(\lambda)$

PHST = one parameter family of random infinite maps of the plane, deformation of the UIPT

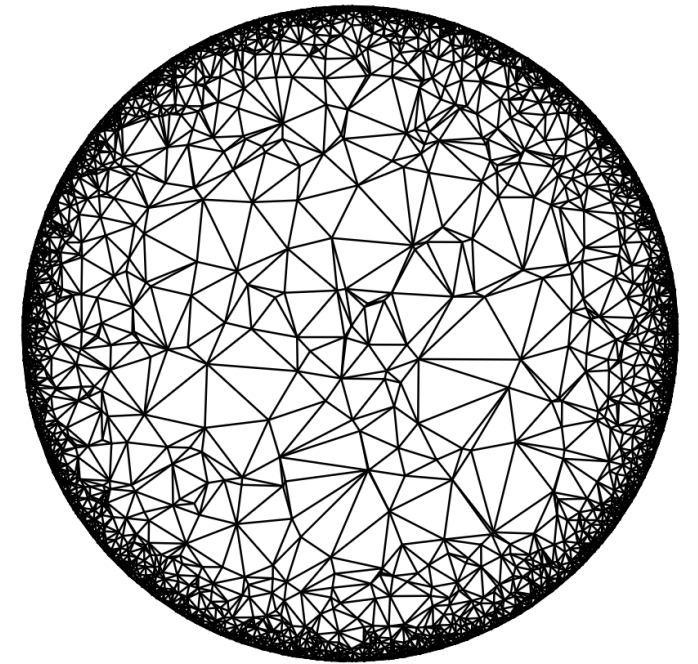


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Hyperbolic features, e.g. $\text{vol}(B_r) = \exp(cr)$

A surprising corollary

For all r constant, $B_r(\mathcal{M}_n)$ is planar with proba $1 - o(1)$!

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Actually ...

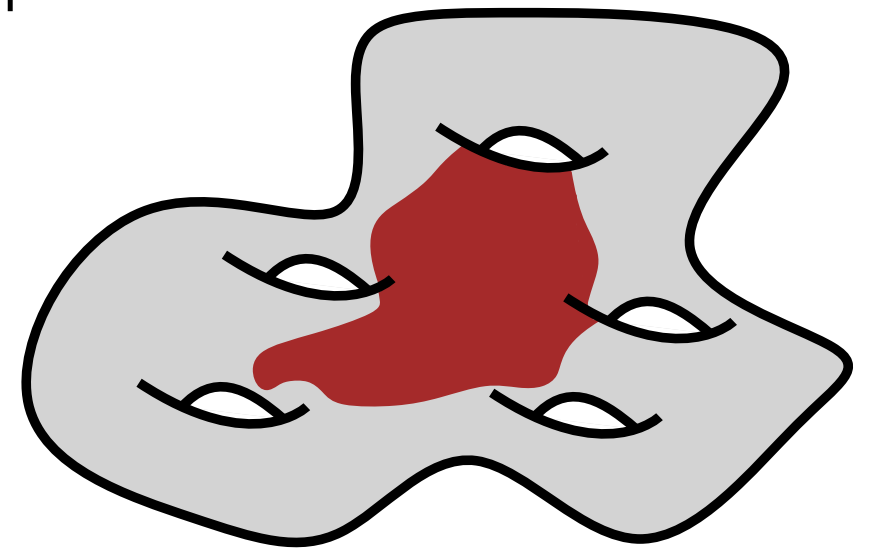
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Actually ...

Theorem [L. 20+]:

There exists a constant a such that if $r = a \log n$, then $B_r(\mathbf{M}_n)$ is planar with proba $1 - o(1)$



Asymptotics

Let $\tau(n, g_n)$ = number of triangulations of genus g_n with $2n$ triangles

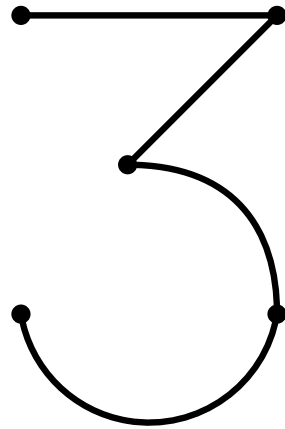
Theorem [Budzinski–L. 19]:

There are explicit continuous functions λ and f such that:

$$\frac{\tau(n-1, g_n)}{\tau(n, g_n)} \rightarrow \lambda(\theta)$$

and

$$\tau(n, g_n) = n^{2g_n} \exp(nf(\theta) + o(n))$$



What's left to discover

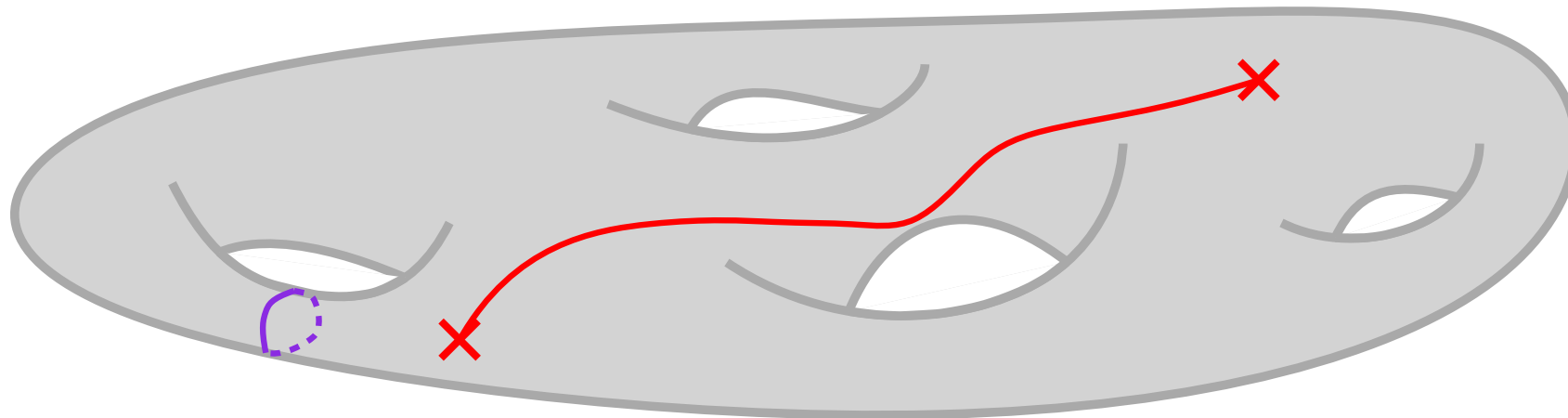
Global observables ?

Conjecture:

The diameter of high genus maps is $\Theta(\log n)$
(lower bound OK).

Conjecture:

The systole (size of smallest non contractible cycle) of high genus maps is a.s. finite).



Asymptotics ?

Open problem:

Find an asymptotic equivalent of $\tau(n, g_n)$

Conjecture (with Elvey-Price–Fang–Wallner):

$$\tau(n, g) \sim t_g n^{\frac{5}{2}(g-1)} \exp\left(n f\left(\frac{g}{n}\right)\right) J\left(\frac{g}{n}\right)$$

Bijections ?

Recurrence formula for triangulations [Goulden–Jackson '08]

$$(n + 1)\tau(n, g) = 4n(3n - 2)(3n - 4)\tau(n - 2, g - 1) + 4(3n - 1)\tau(n - 1, g) + 4 \sum_{i+j=n-2} \sum_{g_1+g_2=g} (3i + 2)(3j + 2)\tau(i, g_1)\tau(j, g_2)$$

Very combinatorial, but proven by algebraic means \rightarrow we want a bijection !

Particular cases known:

- triangulations with only one vertex [Chapuy–Féray–Fusy 14]
- planar triangulations [L. 19], reinterpreted with **lambda-terms** [Singh 22]

Bijections often help understand geometric properties of random maps !

Other geometric models ?

Hope: our geometric/combinatorial/enumeration methods for maps could be useful for other models of maps in high genus !

- hyperbolic surfaces (with or without cusps)
- square tiled surfaces
- Hurwitz numbers (branched covers of the sphere)
- ...

MERCI!