Some questions around high genus maps

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image : N. Curien
Maps: definitions and history
Definition: maps

Map = gluing of polygons along their edges to create a (compact, connected, oriented) surface
Genus $g$ of the map = genus of the surface = # of handles
Rooted = an oriented edge is distinguished
Definition: maps

Alternative definition = graphs embedded on surfaces
Definition: maps

Alternative definition = graphs embedded on surfaces

(alternative definition = factorisations of permutations)
Large random maps

Maps = good model of discrete geometry
→ what does a large random map look like?
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Family $\mathcal{M}_n$ of random maps parametrized by size $n$.
Take $M_n \in \mathcal{M}_n$ uar, study its properties/limit as $n \to \infty$
Large random maps

Example: $\mathcal{M}_n =$ planar (genus 0) maps with $n$ edges
Large random maps

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$\rightarrow$ diameter $= \Theta(n^{1/4})$ [Chassaing–Schaeffer ’02]

$\rightarrow$ ”scaling limit” $= \text{Brownian map}$ [Le Gall, Miermont ’13]

image: J. Bettinelli
Large random maps

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Fundamental tool: Schaeffer’s bijection
Planar maps $\leftrightarrow$ labeled trees
Local convergence

Local convergence = limit law of finite neighborhoods of the root

\[ \mathcal{M}_n \to \mathcal{M} \] iff for all finite \( m \):

\[ \mathbb{P}(m \subset \mathcal{M}_n) \to \mathbb{P}(m \subset \mathcal{M}) \]
Local convergence

Local convergence = limit law of finite neighborhoods of the root

\[ M_n \rightarrow M \iff \text{for all finite } m : \]
\[ P(m \subset M_n) \rightarrow P(m \subset M) \]

\( M_n = \) triangulations (all faces of degree 3) of the sphere with \( 2n \) triangles

[Angel, Schramm ’02] : the sequence \( M_n \) converges to the Uniform Infinite Planar Triangulation (UIPT).
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High genus maps
What is a high genus map?

Take \((g_n)_{n \in \mathbb{N}}\) s.t. \(\frac{g_n}{n} \to \theta \in (0, 1/2)\)

For instance, \(\mathcal{M}_n = \) triangulations of genus \(g_n\) with \(2n\) triangles
What is a high genus map?

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Genus goes to infinity very fast!

Average degree of a vertex = \(\frac{6}{1-2\theta}\)

→ related to average curvature

→ expected hyperbolic behaviour
"Toy model": one faced maps

For $\mathcal{M}_n =$ maps with one face, genus $g_n$ and $n$ edges

→ local limit = "hyperbolic tree" (supercritical GW) [Angel–Chapuy–Curien–Ray 15]
→ diameter = $\Theta(\log n)$ [Ray 15]
→ short cycles follow a Poisson law [Janson–L. 21]
→ large expander subgraphs [L. 21]
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Fundamental tool = bijection between one face maps and decorated trees [Chapuy–Féray–Fusy 14]
The local limit of high genus triangulations

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**Conjecture** [Benjamini, Curien], **Theorem** [Budzinski, L. 19]:
Local limit of \(\mathcal{M}_n = \) PSHT(\(\lambda\))

PHST = one parameter family of random infinite
maps of the plane, deformation of the UIPT

Hyperbolic features, e.g. \(\text{vol}(B_r) = \exp(cr)\)
A surprising corollary

For all $r$ constant, $B_r(M_n)$ is planar with proba $1 - o(1)$!
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Actually . . .

**Theorem [L. 20+]:**
There exists a constant $a$ such that if $r = a \log n$, then $B_r(M_n)$ is planar with proba $1 - o(1)$
Asymptotics

Let $\tau(n, g_n) =$ number of triangulations of genus $g_n$ with $2n$ triangles

Theorem [Budzinski–L. 19]:
There are explicit continuous functions $\lambda$ and $f$ such that:

$$\frac{\tau(n-1, g_n)}{\tau(n, g_n)} \to \lambda(\theta)$$

and

$$\tau(n, g_n) = n^{2g_n} \exp(n f(\theta) + o(n))$$
What’s left to discover
Global observables?

**Conjecture:**
The diameter of high genus maps is $\Theta(\log n)$
(lower bound OK).

**Conjecture:**
The systole (size of smallest non contractible cycle) of high
 genus maps is a.s. finite).
Asymptotics?

Open problem:
Find an asymptotic equivalent of \( \tau(n, g_n) \)

Conjecture (with Elvey-Price–Fang–Wallner):

\[
\tau(n, g) \sim t g n^{\frac{5}{2}(g-1)} \exp(n f\left(\frac{g}{n}\right)) J\left(\frac{g}{n}\right)
\]
**Bijections ?**

Recurrence formula for triangulations [Goulden–Jackson '08]

\[(n + 1)\tau(n, g) = 4n(3n - 2)(3n - 4)\tau(n - 2, g - 1) + 4(3n - 1)\tau(n - 1, g) + 4\sum_{i+j=n-2} \sum_{g_1+g_2=g} (3i + 2)(3j + 2)\tau(i, g_1)\tau(j, g_2)\]

Very combinatorial, but proven by algebraic means → we want a bijection !

Particular cases known:
- triangulations with only one vertex [Chapuy–Féray–Fusy 14]
- planar triangulations [L. 19], reinterpreted with lambda-terms [Singh 22]

Bijections often help understand geometric properties of random maps !
Other geometric models?

**Hope:** our geometric/combinatorial/enumeration methods for maps could be useful for other models of maps in high genus!

- hyperbolic surfaces (with or without cusps)
- square tiled surfaces
- Hurwitz numbers (branched covers of the sphere)
- ...
MERCI!