Block gluing distance in Hom shifts
Walk reconfiguration distance in graphs

Silvère Gangloff, Benjamin Hellouin de Menibus, Piotr Opocha

LISN, Université Paris Saclay
GT Combinatoire et Interactions, Bordeaux

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Ioannis (Yannis) Manoussakis
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Subshifts

\( \mathcal{A} \) a finite alphabet \((\square, ■) \text{ and later } a, b)\);
\( \mathbb{Z}^2 \) the grid to be coloured;
\( \mathcal{A}^\ast \) the finite patterns;
\( \mathcal{A}^{\mathbb{Z}^d} \) the infinite configurations.

\( d = 2, \mathcal{F} = \{■■; ■\} \):

Subshift of finite type (SFT): the set of configurations avoiding a finite set of forbidden patterns.
Hom shifts

Let $G$ be an undirected graph. The **Hom shift** $X_G$ is the set of morphisms $\mathbb{Z}^2 \to G$.

Hom shifts are subshifts with **adjacency constraints** and **invariant by rotation and symmetry**.

ex: $X_{K_k}$ are the $k$-colourings of $\mathbb{Z}^2$. 
Why do we like Hom shifts?

\[ G = \begin{array}{c}
\square \\
\text{\hspace{0.5cm}}
\end{array} \begin{array}{c}
\mathbb{C} \\
\text{\hspace{0.5cm}}
\end{array} \begin{array}{c}
\bullet
\end{array} \hspace{2cm} \rightarrow \hspace{2cm} X_G = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array} \]

- Natural definition (invariance by isometry) with a physical meaning;
- Well-studied examples (hard square, colourings, square ice);
- No embedding of computation $\rightarrow$ no (known) undecidable problems;
- Highly nontrivial problems and rich combinatorics.
Take two valid $n \times n$ patterns. Can we glue them together and complete them into a valid configuration?
Block-gluing distance in Hom shifts

Take two valid $n \times n$ patterns. If they are far enough, can we glue them together and complete them into a valid configuration?

$\gamma_G(n)$ is the minimum distance such that this is possible for all $n \times n$ patterns.
Two walks $x$ and $y$ of length $n$ in $G$ are at distance 1 if $x_i$ and $y_i$ are neighbours in $G$ for every $i$.

$x = \begin{array}{cccccccccc}
  x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \\
y = \begin{array}{cccccccccc}
  y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10}
\end{array}
\end{array}$

Walk reconfiguration distance

$\gamma_G(n)$ is the maximal distance between two walks of length $n$ in $G$.

⚠️ $n$ is not the size of the graph!

Claim: it is the same $\gamma_G$ as before.
Warmup

Theorem

For any connected graph $G$, $\gamma_G(n) = O(n)$.
For any connected graph $G$, $\gamma_G(n) = O(n)$.

For any walk $u_0 - a - b - v_4$ in $G$. 

\begin{align*}
&\begin{array}{cccc}
u_0 & u_1 & u_2 & u_3 & u_4 \\
a & u_0 & u_1 & u_2 & u_3 \\
\end{array}
&\begin{array}{cccc}
v_0 & v_1 & v_2 & v_3 & v_4 \\
\end{array}
\end{align*}
Theorem

For any connected graph $G$, $\gamma_G(n) = O(n)$.

for any walk $u_0 - a - b - v_4$ in $G$. 
Part II

The square-free case: Chandgotia & Marcus
Theorem

When $G$ is a tree, $\gamma_G(n) = O(1)$. 

![Tree diagram]

Every walk is at constant distance from a trivial walk.

Still works if $G$ has one loop.
Theorem

When $G$ is a tree, $\gamma_G(n) = O(1)$.

Step 1: shift right.
Theorem

When $G$ is a tree, $\gamma_G(n) = O(1)$.

Step 1: shift right and replace $a \rightarrow c$, $f \rightarrow c$, $d \rightarrow b$. 
**Theorem**
When $G$ is a tree, $\gamma_G(n) = O(1)$.

**Step 1:** shift right and replace $a \rightarrow c$, $f \rightarrow c$, $d \rightarrow b$.

**Step 2:** shift left and replace $e \rightarrow b$. 

Every walk is at constant distance from a trivial walk. 

Still works if $G$ has one loop.
Theorem

When $G$ is a tree, $\gamma_G(n) = O(1)$.

**Step 1:** shift right and replace $a \to c$, $f \to c$, $d \to b$.

**Step 2:** shift left and replace $e \to b$.

Every walk is at constant distance from a trivial walk.

still works if $G$ has one loop.
Characterisation for the square-free case

Theorem (Chandgotia, Marcus 18)

If $G$ is square-free, then $\gamma_G = \Theta(n)$ or $\Theta(1)$. The latter is when $G$ is a tree with $\leq 1$ loop.

**Square-free**: if $x_0 - x_1 - x_2 - x_3 - x_0$, then $x_3 = x_1$ or $x_0 = x_2$. 
Universal cover $\mathcal{U}_G$

Smallest tree with a surjective morphism $\mathcal{U}_G \rightarrow G$
(that “has the same neighbourhoods”).

\[ \rightarrow \quad a \quad b \quad c \quad a \quad b \quad c \quad a \quad b \quad c \quad \ldots \]
Universal cover $\mathcal{U}_G$

Smallest tree with a surjective morphism $\mathcal{U}_G \to G$ (that “has the same neighbourhoods”).

Concrete construction

Choose an arbitrary vertex in $G$ (say, $a$).

Vertices of $\mathcal{U}_G$ correspond to walks with no backtrack in $G$ starting from $a$. 

\[
\begin{array}{cccccccc}
  & a & b & c & a & b & c & a & b & c \\
-3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \ldots
\end{array}
\]
Universal cover $\mathcal{U}_G$

Smallest tree with a surjective morphism $\mathcal{U}_G \to G$ (that “has the same neighbourhoods”).

Concrete construction

Choose an arbitrary vertex in $G$ (say, $a$).

Vertices of $\mathcal{U}_G$ correspond to walks with no backtrack in $G$ starting from $a$. 
Lifting

From $G$ to its universal cover

If $G$ is square-free, any colouring $x \in X_G$ lifts to a colouring of $X_{U_G}$ that maps to $x$.

Assign the empty walk (0) to any point, then choose the only possibility.

\[
\begin{array}{cccccc}
  a & b & c & a & b & c \\
  -3 & -2 & -1 & 0 & 1 & 2 \\
\end{array}
\]

\[
\begin{array}{cccccc}
  a & b & a & c & b & a \\
  b & c & b & a & c & b \\
  c & b & c & b & a & c \\
  a & c & a & c & b & a \\
  b & a & c & b & a & c \\
\end{array}
\]

\[
\begin{array}{cccccc}
  0 & 1 & 0 & -1 & -2 \\
  1 & 2 & 1 & 0 & -1 \\
  2 & 1 & 2 & 1 & 0 \\
  3 & 2 & 3 & 2 & 1 \\
  4 & 3 & 2 & 1 & 0 \\
\end{array}
\]
Lifting

Why is this working?

Why can’t two paths with the same origin and destination reach different vertices in $U_G$?

If $G$ is square-free, then $x_3 = x_1$ or $x_0 = x_2$.

$\rightarrow x_0 x_1 x_2 x_3 x_0$ is a **backtracking** (= trivial) cycle.
Why is this working?

Why can’t two paths with the same origin and destination reach different vertices in $\mathcal{U}_G$?

If $G$ is square-free, every cycle in $X_G$ is a **backtracking** (= trivial) cycle.
The square-free case

Theorem (Chandgotia, Marcus 18)

If $G$ is square-free and $|\mathcal{U}_G| = +\infty$, then $\gamma_G = \Theta(n)$.
The square-free case

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If $G$ is square-free and $|\mathcal{U}_G| = +\infty$, then $\gamma_G = \Theta(n)$.

Distance between the “furthest possible” walk and a trivial walk.

\[
\begin{array}{c c c}
\text{a} & \text{b} & \text{c} \\
\text{a} & \text{b} & \text{c} \\
\hdashline
\text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} \\
\end{array}
\quad
da
\quad
\begin{array}{c c c c c c c}
0 & 1 & 2 & 3 & 4 & 5 \\
\text{k} & \text{k} \pm 1 & \text{k} & \text{k} \pm 1 & \text{k} & \text{k} \pm 1 \\
\end{array}
\]

\[d \leq n \leq d\]
The square-free case

Theorem (Chandgotia, Marcus 18)

If $G$ is square-free and $|\mathcal{U}_G| = +\infty$, then $\gamma_G = \Theta(n)$.

Distance between the “furthest possible” walk and a trivial walk.

$k$ is at distance $\leq d$ from 0 and $n$ (in $\mathcal{U}_G$) so $d \geq n/2$. 
The square-free case

Theorem (Chandgotia, Marcus 18)

If $G$ is square-free and $|\mathcal{U}_G| = +\infty$, then $\gamma_G = \Theta(n)$.

Two behaviours for $\gamma_G$ in square-free graphs:

- $\Theta(1)$ if $G$ is a tree with up to one loop;
- $\Theta(n)$ in general.

Pavlov and Marcus conjectured that these are the only possible behaviours.
Part III

The squareful case: our contribution
If $G$ has squares, lifting to the universal cover fails.
The square cover

Square cover of $G$

$U_G^\square$ is obtained from $U_G$ by identifying vertices that are equal up to a square.
The square cover

Square cover of $G$

$\mathcal{U}_G^{\square}$ is obtained from $\mathcal{U}_G$ by identifying vertices that are equal up to a square.
The square cover

Square cover of $G$

$\mathcal{U}_G^{\square}$ is obtained from $\mathcal{U}_G$ by identifying vertices that are equal up to a square.
The square cover

From $G$ to its universal cover

If $G$ is square-free, any colouring $x \in X_G$ lifts to a colouring of $X_{U_G}$ that maps to $x$.

Theorem (Chandgotia, Marcus 18)

If $G$ is square-free and $|U_G| = +\infty$, then $\gamma_G = \Theta(n)$.

Theorem (Gangloff, H., Opocha 22)

If $|U_{\Box G}| = +\infty$, then $\gamma_G = \Theta(n)$.

What happens when $|U_{\Box G}| < +\infty$?
The square cover

From $G$ to its universal cover
If $G$ is square-free, any colouring $x \in X_G$ lifts to a colouring of $X_{\mathcal{U}_G}$ that maps to $x$.

From $G$ to its square cover
Any colouring $x \in X_G$ lifts to a colouring of $X_{\mathcal{U}_G^\square}$ that maps to $x$.

Theorem (Chandgotia, Marcus 18)
If $G$ is square-free and $|\mathcal{U}_G| = +\infty$, then $\gamma_G = \Theta(n)$.

Theorem (Gangloff, H., Opocha 22)
If $|\mathcal{U}_G^\square| = +\infty$, then $\gamma_G = \Theta(n)$.

What happens when $|\mathcal{U}_G^\square| < +\infty$?
A worst-case logarithmic bound

Theorem (Gangloff, H., Opocha 22)

If $|\mathcal{U}_G| < +\infty$, then $\gamma_G = O(\log n)$.

Lift to $\mathcal{U}_G$ and consider a cycle $c^n$.

$c$ can be decomposed into squares in the following sense:

(i) $c_i$

(ii) $c_i \  c_{i+1}$

(iii) $c_i \  c_{i+1}$
A worst-case logarithmic bound

**Theorem (Gangloff, H., Opocha 22)**

If $|\mathcal{U}_G^\square| < +\infty$, then $\gamma_G = O(\log n)$.

Lift to $\mathcal{U}_G^\square$ and consider a cycle $c^n$. $c$ can be decomposed into squares in the following sense:
A worst-case logarithmic bound

Theorem (Gangloff, H., Opocha 22)

If $|\mathcal{U}_G^\square| < +\infty$, then $\gamma_G = O(\log n)$.

Lift to $\mathcal{U}_G^\square$ and consider a cycle $c^n$.

$c$ can be decomposed into squares in the following sense:

(ii)

(c_{0} c_{1} c_{2} c_{3}) (c_{0} c_{1} c_{2} c_{3} c_{0} c_{1} c_{2} c_{3}) (c_{0} c_{1} c_{2} c_{3})
A worst-case logarithmic bound

Theorem (Gangloff, H., Opocha 22)

If $|\mathcal{U}_G^\square| < +\infty$, then $\gamma_G = O(\log n)$.

Lift to $\mathcal{U}_G^\square$ and consider a cycle $c^n$.

$c$ can be decomposed into squares in the following sense:

$$(iii)$$

\begin{array}{|c|c|c|c|}
\hline
 c_0 & c_1 & c_2 & c_3 \\
\hline
 c_1 & c_2 & c_3 & c_0 \\
\hline
 c_1 & a & b & c_2 \\
\hline
 c_1 & c_2 & c_3 & c_0 \\
\hline
\end{array}$
Theorem (Gangloff, H., Opocha 22)

If $|\mathcal{U}_G^\square| < +\infty$, then $\gamma_G = O(\log n)$.

Lift to $\mathcal{U}_G^\square$ and consider a cycle $c^n$. $c$ can be decomposed into squares in the following sense:
The Ken-no-Katabami graph

(Thanks to Jan van der Heuvel)
The Ken-no-Katabami graph

Theorem (Gangloff, H., Opocha 22)

\[ \gamma_G(n) = \Theta(\log n) \] for the Ken-no-Katabami graph.

Critical walk: \((abcdef)^n\) is at distance \(\log n\) from a trivial cycle.

Key property: cycles at distance 1 from \(abcdef\) are all larger, so nothing can be done in parallel.

(Thanks to Jan van der Heuvel)
Conclusion

There are three possible behaviours for $\gamma_G$:

- $\Theta(n)$: infinite square cover.
- $\Theta(1)$: every cycle can be square-decomposed through smaller cycles.
- $\Theta(\log n)$: some cycle cannot be square-decomposed in this way.

Open questions:

- Intermediate behaviour between $\Theta(\log n)$ – $\Theta(1)$?
- What about higher-dimensional Hom shifts? Is there a logarithmic case?

Work in progress (with Chandgotia, Gangloff, Oprocha)

The cases $\Theta(n)$ and $O(\log n)$ are computably unseparable.