

# Block gluing distance in Hom shifts

## Walk reconfiguration distance in graphs

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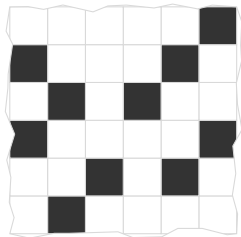


Dédié à la mémoire de  
**Ioannis (Yannis) Manoussakis**  
décédé le 5 juin 2021

# Subshifts

- $\mathcal{A}$  a finite **alphabet** ( $\square, \blacksquare$  and later  $a, b$ );
- $\mathbb{Z}^2$  the **grid** to be coloured;
- $\mathcal{A}^*$  the finite **patterns**;
- $\mathcal{A}^{\mathbb{Z}^d}$  the infinite **configurations**.

$$d = 2, \mathcal{F} = \{\blacksquare\blacksquare; \blacksquare\blacksquare\}:$$



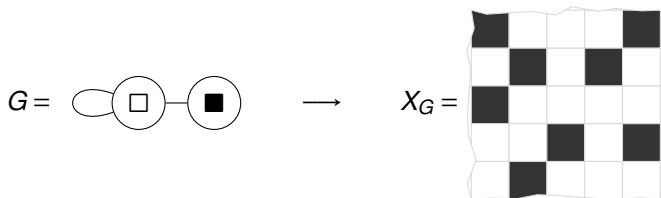
**Subshift of finite type (SFT):** the set of configurations avoiding a finite set of forbidden patterns.

# Hom shifts

## Hom shift

Let  $G$  be an undirected graph.

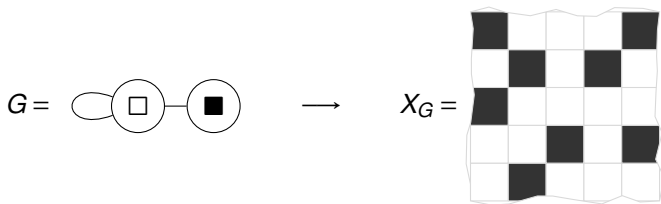
The **Hom shift**  $X_G$  is the set of morphisms  $\mathbb{Z}^2 \rightarrow G$ .



Hom shifts are subshifts with **adjacency constraints** and **invariant by rotation and symmetry**.

ex:  $X_{K_k}$  are the  $k$ -colourings of  $\mathbb{Z}^2$ .

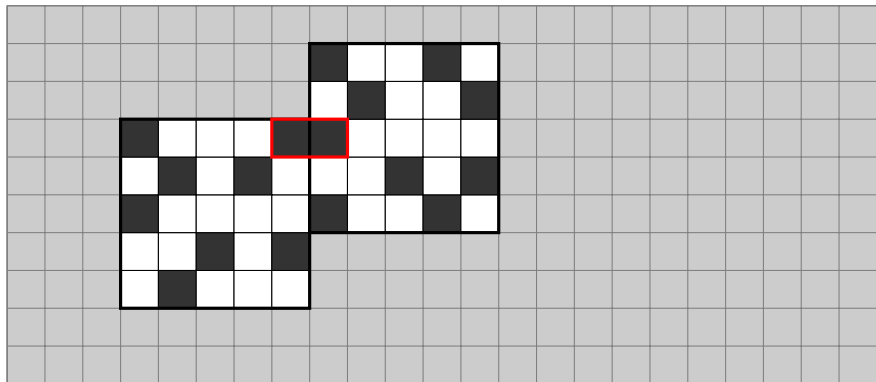
## Why do we like Hom shifts?



- ▶ Natural definition (invariance by isometry) with a physical meaning;
- ▶ Well-studied examples (hard square, colourings, square ice);
- ▶ No embedding of computation  $\rightarrow$  no (known) undecidable problems;
- ▶ Highly nontrivial problems and rich combinatorics.

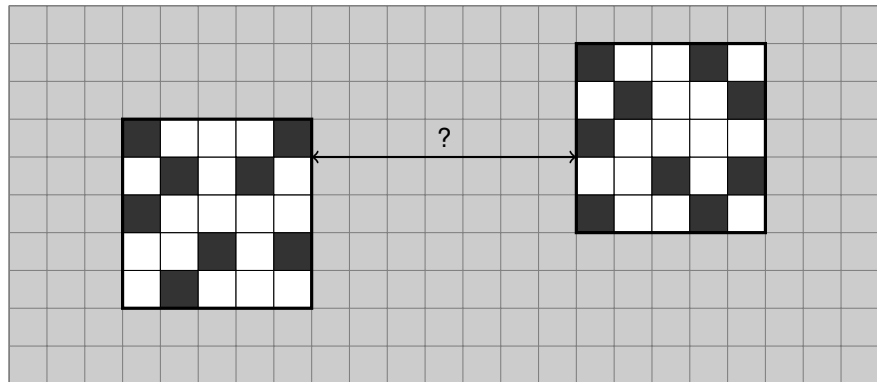
## Block-gluing distance in Hom shifts

Take two valid  $n \times n$  patterns. Can we glue them together and complete them into a valid configuration?



## Block-gluing distance in Hom shifts

Take two valid  $n \times n$  patterns. If they are far enough, can we glue them together and complete them into a valid configuration?



### Block-gluing distance

$\gamma_G(n)$  is the minimum distance such that this is possible for all  $n \times n$  patterns.

## Walk reconfiguration distance in graphs

Two walks  $x$  and  $y$  of length  $n$  in  $G$  are **at distance 1** if  $x_i$  and  $y_i$  are neighbours in  $G$  for every  $i$ .

$x =$	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$x_9$	$x_{10}$
$y =$	$y_0$	$y_1$	$y_2$	$y_3$	$y_4$	$y_5$	$y_6$	$y_7$	$y_8$	$y_9$	$y_{10}$

### Walk reconfiguration distance

$\gamma_G(n)$  is the maximal distance between two walks of length  $n$  in  $G$ .

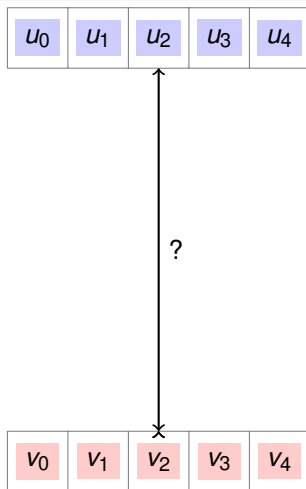
⚠  $n$  is not the size of the graph!

Claim: it is the same  $\gamma_G$  as before.

# Warmup

## Theorem

For any connected graph  $G$ ,  $\gamma_G(n) = O(n)$ .





# Warmup

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For any connected graph  $G$ ,  $\gamma_G(n) = O(n)$ .

$u_0$	$u_1$	$u_2$	$u_3$	$u_4$
a	$u_0$	$u_1$	$u_2$	$u_3$

for any walk  
 $u_0 - a - b - v_4$  in  $G$ .

$v_0$	$v_1$	$v_2$	$v_3$	$v_4$
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# Warmup

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a	$u_0$	$u_1$	$u_2$	$u_3$
b	a	$u_0$	$u_1$	$u_2$
$v_4$	b	a	$u_0$	$u_1$
$v_3$	$v_4$	b	a	$u_0$
$v_2$	$v_3$	$v_4$	b	a
$v_1$	$v_2$	$v_3$	$v_4$	b
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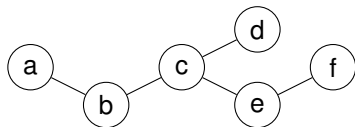
## Part II

# The square-free case: Chandgotia & Marcus

# Trees

## Theorem

When  $G$  is a tree,  $\gamma_G(n) = O(1)$ .

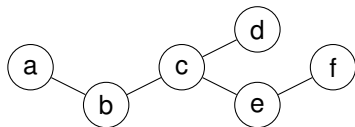


a	b	c	e	f	e	c	d	c	b
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# Trees

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↙

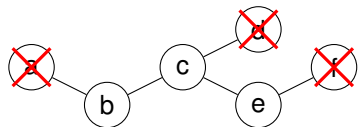
a	b	c	e	f	e	c	d	c	b
b	a	b	c	e	f	e	c	d	c

**Step 1:** shift right.

# Trees

## Theorem

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↙

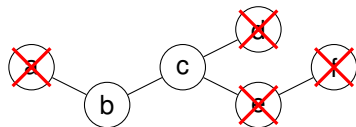
a	b	c	e	f	e	c	d	c	b
b	<b>c</b>	b	c	e	<b>c</b>	e	c	<b>b</b>	c

**Step 1:** shift right and replace  $a \rightarrow c$ ,  $f \rightarrow c$ ,  $d \rightarrow b$ .

# Trees

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a	b	c	e	f	e	c	d	c	b
b	<b>c</b>	b	c	e	<b>c</b>	e	c	<b>b</b>	c
c	b	c	<b>b</b>	c	<b>b</b>	c	b	c	b

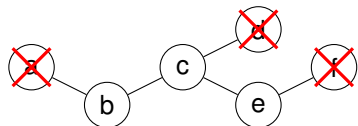
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**Step 2:** shift left and replace  $e \rightarrow b$ .

# Trees

## Theorem

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a	b	c	e	f	e	c	d	c	b
b	<b>c</b>	b	c	e	<b>c</b>	e	c	<b>b</b>	c
c	b	c	<b>b</b>	c	<b>b</b>	c	b	c	b

**Step 1:** shift right and replace  $a \rightarrow c$ ,  $f \rightarrow c$ ,  $d \rightarrow b$ .

**Step 2:** shift left and replace  $e \rightarrow b$ .

Every walk is at constant distance from a trivial walk.

still works if  $G$  has one loop.



## Characterisation for the square-free case

### Theorem (Chandgotia, Marcus 18)

If  $G$  is square-free, then  $\gamma_G = \Theta(n)$  or  $\Theta(1)$ .

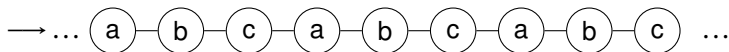
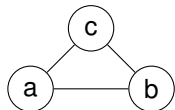
The latter is when  $G$  is a tree with  $\leq 1$  loop.

**Square-free** : if  $x_0 - x_1 - x_2 - x_3 - x_0$ , then  $x_3 = x_1$  or  $x_0 = x_2$ .

# Universal cover

## Universal cover $\mathcal{U}_G$

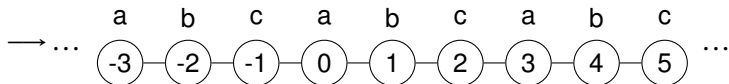
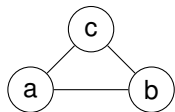
Smallest tree with a surjective morphism  $\mathcal{U}_G \rightarrow G$   
(that “has the same neighbourhoods”).



# Universal cover

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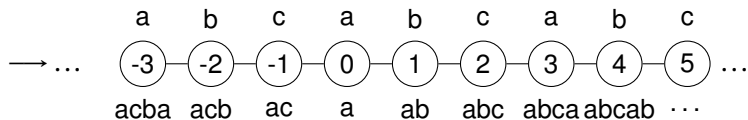
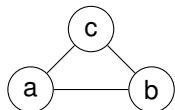
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# Universal cover

## Universal cover $\mathcal{U}_G$

Smallest tree with a surjective morphism  $\mathcal{U}_G \rightarrow G$   
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## Concrete construction

Choose an arbitrary vertex in  $G$  (say,  $a$ ).

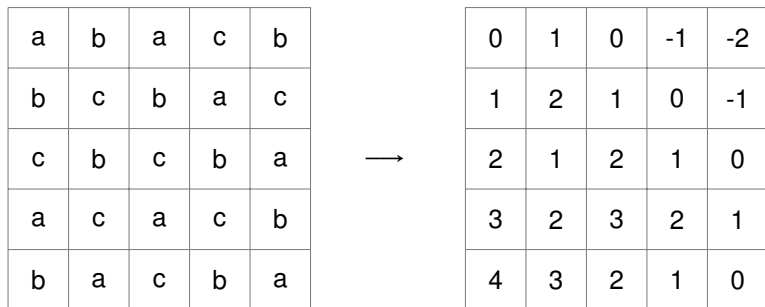
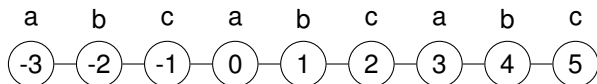
Vertices of  $\mathcal{U}_G$  correspond to **walks with no backtrack** in  $G$  starting from  $a$ .

# Lifting

## From $G$ to its universal cover

If  $G$  is square-free, any colouring  $x \in X_G$  lifts to a colouring of  $X_{\mathcal{U}_G}$  that maps to  $x$ .

Assign the empty walk (0) to any point, then choose the only possibility.



## Why is this working?

Why can't two paths with the same origin and destination reach different vertices in  $\mathcal{U}_G$ ?

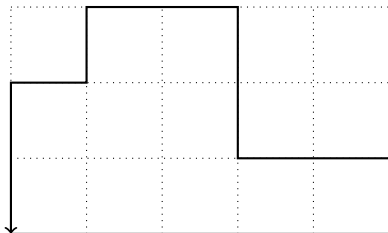
$x_3$	$x_2$
$x_0$	$x_1$

If  $G$  is square-free, then  $x_3 = x_1$  or  $x_0 = x_2$ .

→  $x_0x_1x_2x_3x_0$  is a **backtracking** (= trivial) cycle.

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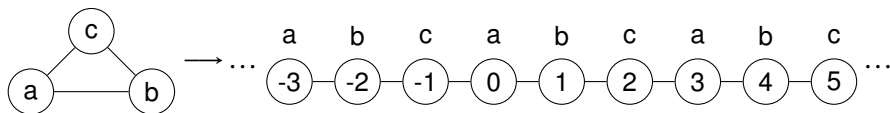


If  $G$  is square-free, every cycle in  $X_G$  is a **backtracking** (= trivial) cycle.

## The square-free case

### Theorem (Chandgotia, Marcus 18)

If  $G$  is square-free and  $|\mathcal{U}_G| = +\infty$ , then  $\gamma_G = \Theta(n)$ .



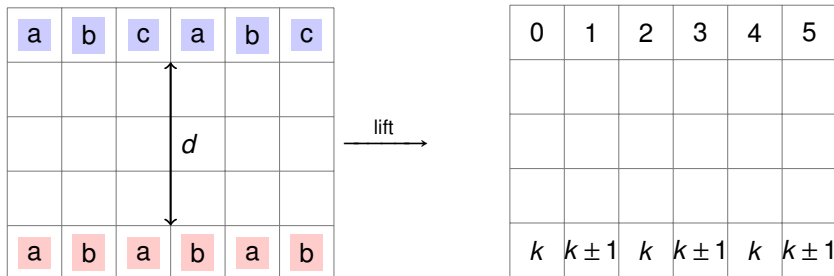


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Distance between the “furthest possible” walk and a trivial walk.

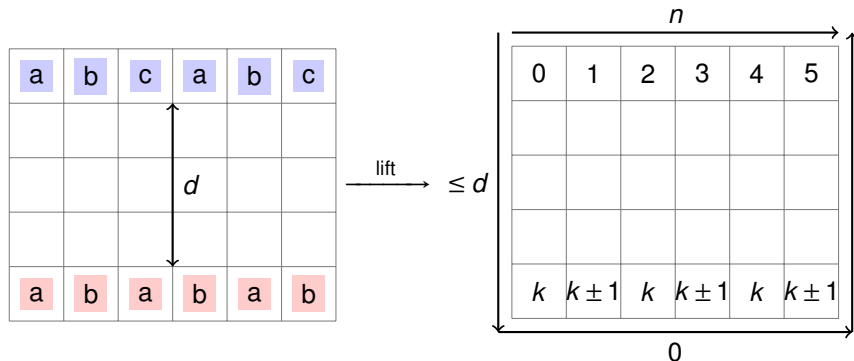


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$k$  is at distance  $\leq d$  from  $0$  and  $n$  (in  $\mathcal{U}_G$ ) so  $d \geq n/2$ .

## The square-free case

### Theorem (Chandgotia, Marcus 18)

If  $G$  is square-free and  $|\mathcal{U}_G| = +\infty$ , then  $\gamma_G = \Theta(n)$ .

Two behaviours for  $\gamma_G$  in square-free graphs:

- ▶  $\Theta(1)$  if  $G$  is a tree with up to one loop;
- ▶  $\Theta(n)$  in general.

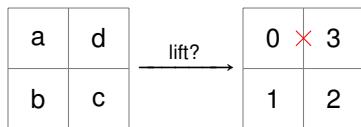
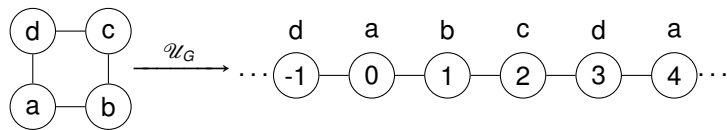
Pavlov and Marcus conjectured that these are the only possible behaviours.

## Part III

### The squareful case: our contribution

## Fixing the universal cover

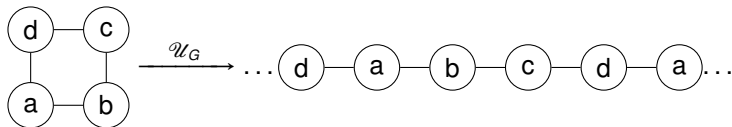
If  $G$  has squares, lifting to the universal cover fails.



# The square cover

## Square cover of $G$

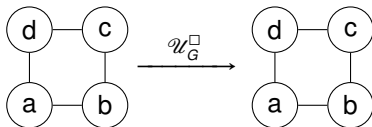
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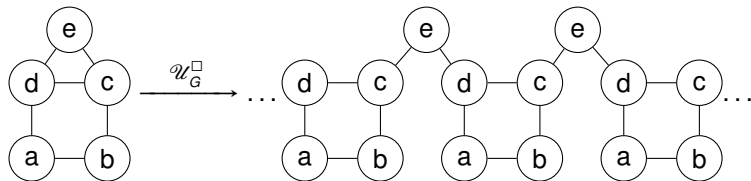
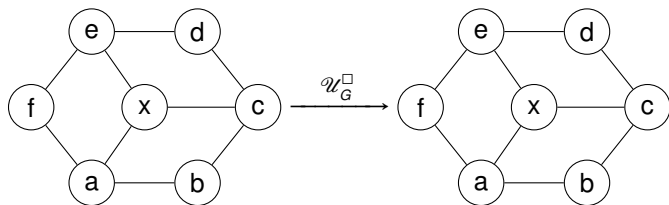
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## The square cover

### From $G$ to its universal cover

If  $G$  is square-free, any colouring  $x \in X_G$  lifts to a colouring of  $X_{\mathcal{U}_G}$  that maps to  $x$ .

### Theorem (Chandgotia, Marcus 18)

If  $G$  is square-free and  $|\mathcal{U}_G| = +\infty$ , then  $\gamma_G = \Theta(n)$ .

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## From $G$ to its square cover

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## Theorem (Gangloff, H., Opocha 22)

If  $|\mathcal{U}_G^\square| = +\infty$ , then  $\gamma_G = \Theta(n)$ .

What happens when  $|\mathcal{U}_G^\square| < +\infty$ ?

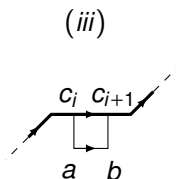
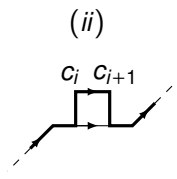
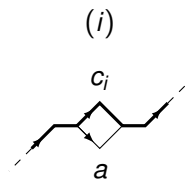
## A worst-case logarithmic bound

Theorem (Gangloff, H., Opocha 22)

If  $|\mathcal{U}_G^\square| < +\infty$ , then  $\gamma_G = O(\log n)$ .

Lift to  $\mathcal{U}_G^\square$  and consider a cycle  $c^n$ .

$c$  can be decomposed into squares in the following sense:



## A worst-case logarithmic bound

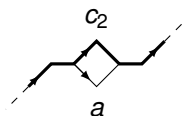
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(i)



$c_0$	$c_1$	$c_2$	$c_3$	$c_0$	$c_1$	$c_2$	$c_3$	$c_0$	$c_1$	$c_2$
$c_1$	$a$	$c_3$	$c_0$	$c_1$	$a$	$c_3$	$c_0$	$c_1$	$a$	$c_3$

## A worst-case logarithmic bound

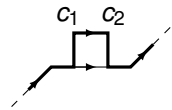
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(ii)



$c_0$	$c_1$	$c_2$	$c_3$	$c_0$	$c_1$	$c_2$	$c_3$	$c_0$	$c_1$	$c_2$
$c_3$	$c_0$	$c_3$	$c_0$	$c_3$	$c_0$	$c_3$	$c_0$	$c_3$	$c_0$	$c_3$

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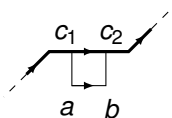
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$c_1$	$c_2$	$c_3$	$c_0$	$c_1$	$a$	$b$	$c_2$	$c_3$	$c_0$	$c_1$

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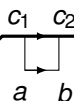
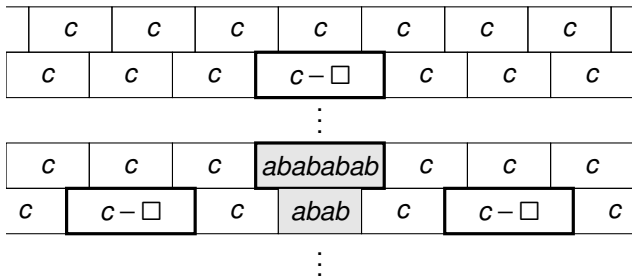
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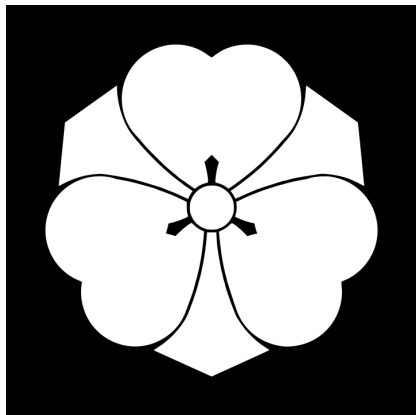
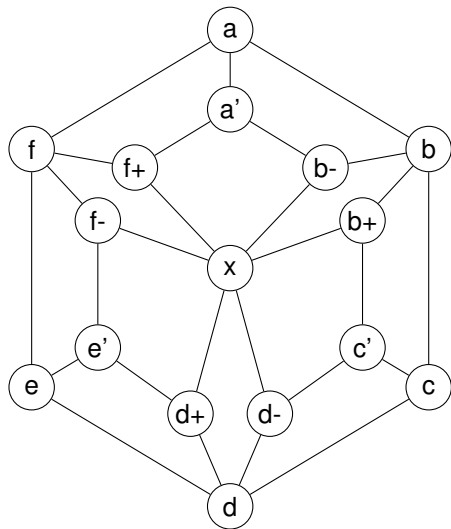
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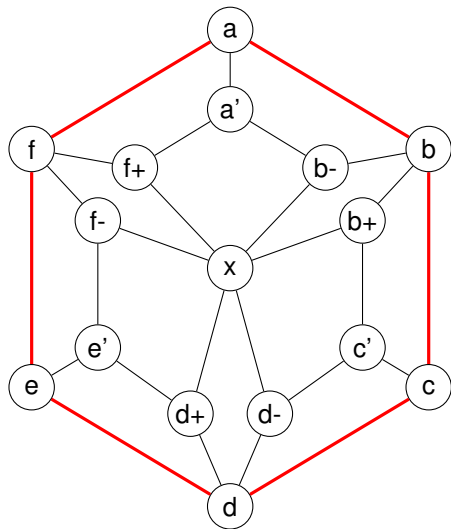
# The Ken-no-Katabami graph



(Thanks to Jan van der Heuvel)



# The Ken-no-Katabami graph



## Theorem

(Gangloff, H., Opocha 22)

$\gamma_G(n) = \Theta(\log n)$  for the Ken-no-Katabami graph.

**Critical walk:**  $(abcdef)^n$  is at distance  $\log n$  from a trivial cycle.

**Key property:** cycles at distance 1 from  $abcdef$  are all **larger**, so nothing can be done in parallel.

(Thanks to Jan van der Heuvel)

# Conclusion

There are three possible behaviours for  $\gamma_G$ :

- ▶  $\Theta(n)$ : infinite square cover.
- ▶  $\Theta(1)$ : every cycle can be square-decomposed through smaller cycles.
- ▶  $\Theta(\log n)$ : some cycle cannot be square-decomposed in this way.

## Open questions:

- ▶ Intermediate behaviour between  $\Theta(\log n) - \Theta(1)$  ?
- ▶ What about higher-dimensional Hom shifts? Is there a logarithmic case?

Work in progress (with Chandgotia, Gangloff, Oprocha)

The cases  $\Theta(n)$  and  $O(\log n)$  are computably unseparable.