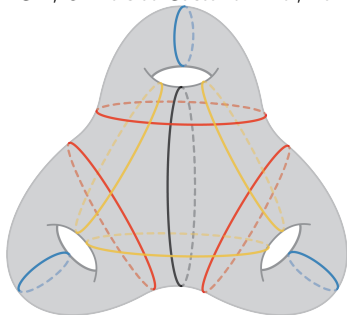


UNIVERSAL FAMILIES OF ARCS AND CURVES ON SURFACES

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Hugo Parlier

LIGM, Université Gustave Eiffel, Paris



Séminaire de combinatoire, LaBRI, May 2023

OUTLINE

1 INTRODUCTION

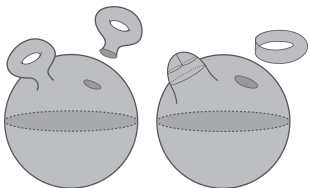
2 A MOTIVATION

3 REALIZING PANTS DECOMPOSITIONS

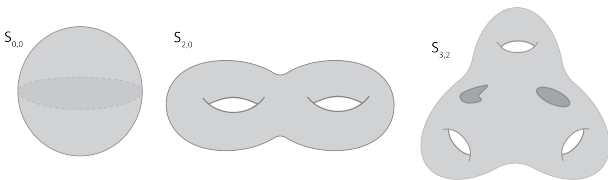
4 REALIZING ALL TRIANGULATIONS

SURFACES

- A **surface** is a topological space that locally looks like a plane or a half plane.

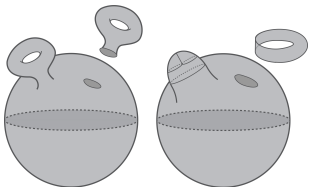


- We deal with **connected**, **compact**, **orientable** surfaces determined by their genus and number of punctures.

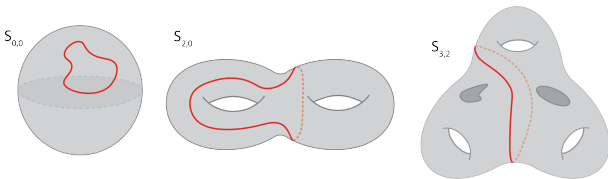


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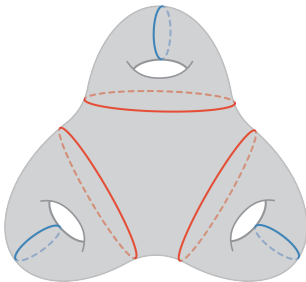


- We deal with **connected**, **compact**, **orientable** surfaces determined by their genus and number of punctures.
- Closed curves are maps from S^1 to the surface. We deal with **simple** curves.



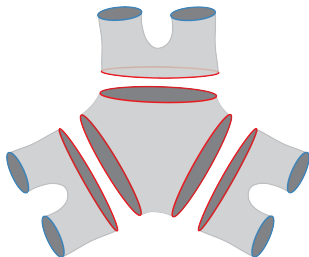
PANTS DECOMPOSITION

- A *pants decomposition* is a set of curves that cuts the surface into pairs of pants



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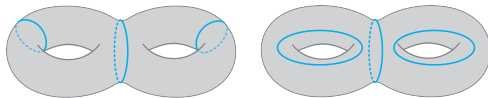
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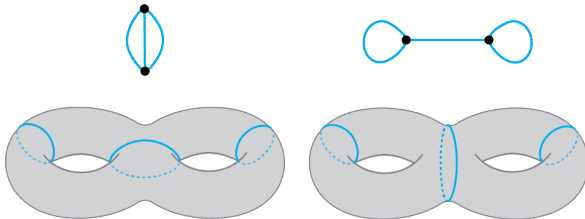
- A surface $S_{g,b}$ is decomposed to $2g - 2 + b$ pairs of pants.
- A pants decomposition of $S_{g,b}$ contains $3g - 3 + b$ curves.

REALIZING ALL PANTS DECOMPOSITIONS

- Two pants decompositions are of the same *homeomorphism type* if there exists a homeomorphism of the surface that maps one to the other.



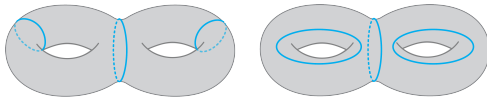
- A dual trivalent graph that encodes the adjacencies of the pants determines the homeomorphism type of a pants decomposition.



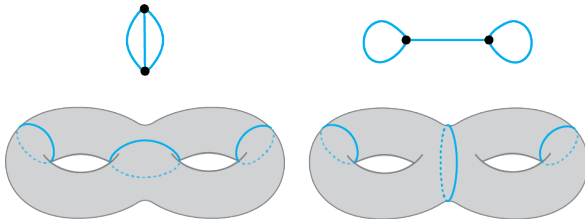
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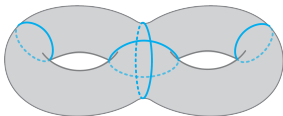
- A *pants universal family of curves* is a set of curves that contains all types of pants decomposition of the surface
- ★ **Problem:** What is the size of this family?

ESTIMATING THE SIZE OF THE UNIVERSAL FAMILY

- A surface of genus g has $g^{\Theta(g)}$ homeomorphism types of pants decompositions.
- There exists an upper bound of $g^{O(g)}$ for the size of such a family.

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- A surface of genus g has $g^{\Theta(g)}$ homeomorphism types of pants decompositions.
- There exists an upper bound of $g^{O(g)}$ for the size of such a family.
- We can improve this upper bound by reusing the curves:



THEOREM (F., DE MESMAY, PARLIER)

Let Γ be a minimal size universal family for pants decompositions of a surface of genus g without punctures. Then

$$|\Gamma| \leq 3^{2g-1} \quad \text{and} \quad |\Gamma| = \Omega(g^{\frac{4}{3}-\epsilon}) \quad \text{for any } \epsilon > 0.$$

Question: Can we close the gap between the upper and lower bound?

REALIZING PANTS DECOMPOSITIONS

Given a sphere with n punctures, we can consider two kinds of homeomorphisms:

- homeomorphisms that fix punctures pointwise (labelled punctures)
- homeomorphisms that can permute the punctures (unlabelled punctures)

REALIZING PANTS DECOMPOSITIONS

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	Lower bound	Upper bound
Surface of genus g without punctures	$\Omega(n^{\frac{4}{3}})$	3^{2g-1}
Sphere with n unlabelled punctures	$\Omega(n \log(n))$	$O(n^2)$
Sphere with n labelled punctures	$2^{n-1} - n - 1$	3^{n-1}

A MOTIVATION: UNIVERSAL SHORTEST PATH METRIC

THEOREM (FÀRY-WAGNER)

All planar graphs can be drawn in the plane with straight lines.

CONJECTURE (HUBARD, KALUŽA, TANCER) '17

For any surface S_g , there exists a universal Riemannian metric (***universal shortest path metric***) such that all graphs embeddable on S_g can be embedded with shortest paths.

- It is true for sphere, the projective plane, torus and the Klein bottle.
- For higher g , most metrics are not a universal shortest path metric.

THEOREM (HUBARD, KALUŽA, DE MESMAY, TANCER '17)

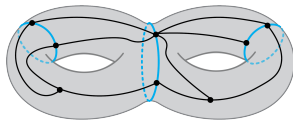
For every $g > 1$, there exists a universal metric for which all graphs embeddable on the surface can be embedded such that edges are concatenations of $O(g)$ shortest paths.

A MOTIVATION: UNIVERSAL SHORTEST PATH METRIC

OBSERVATION

A super-polynomial lower bound on the size of a pants universal family of curves leads to a negation of the universal shortest metric conjecture.

- For any pants decomposition of the surface create a graph embedding that contains the decomposition
- Assuming the conjecture, all these graphs have a shortest path embedding
- each curve in the pants decomposition embeds with constant number of shortest paths
- Each pair of curves in the family cross constantly

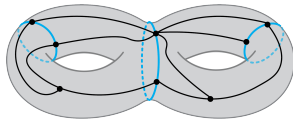


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- each curve in the pants decomposition embeds with constant number of shortest paths
- Each pair of curves in the family cross constantly
- Maximal number of non-homotopic simple closed curves on a surface that cross constantly is polynomial in g . [Greene '18]*



THE CASE OF SURFACES WITHOUT PUNCTURES

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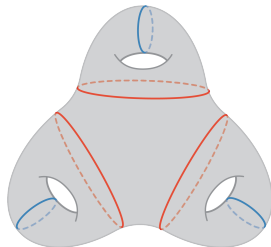
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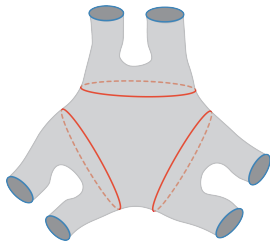
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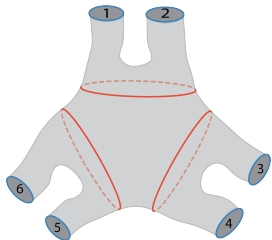
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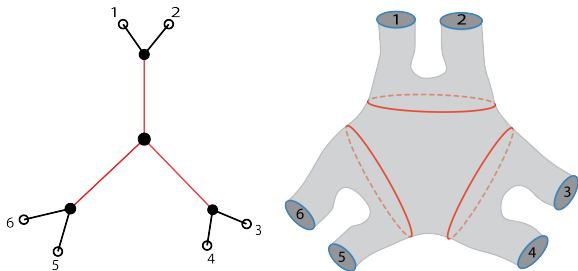
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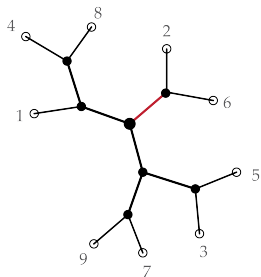
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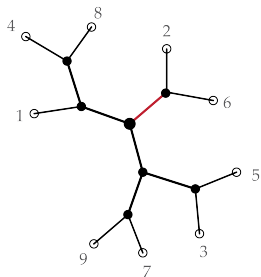
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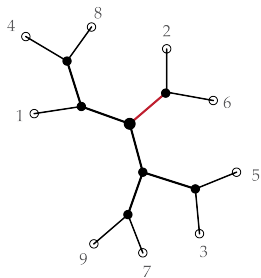
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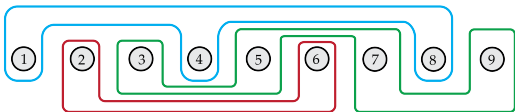
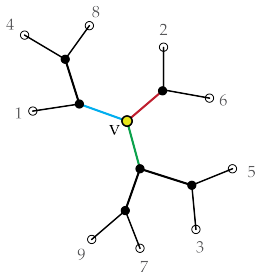


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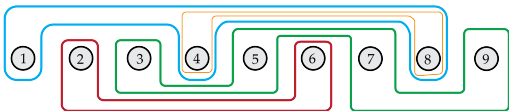
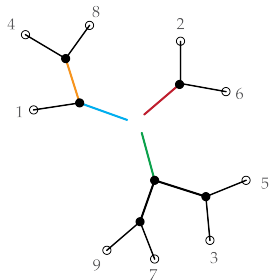


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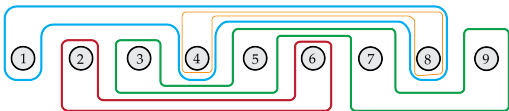
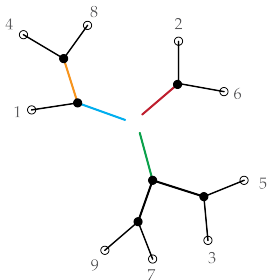


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- Claim: The family of all these wiggling separating curves realize all the trivalent trees.
- The size of this family is less than 3^{n-1} .

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→ Lower bound:

- There exists up to g^g types of pants decompositions.
- Each pants decomposition consists of $3g - 3$ curves.

$$\binom{|\Gamma|}{3g-3} \gtrsim g^g \rightarrow |\Gamma| = \Omega(g^{\frac{4}{3}-\epsilon})$$

Question: Can we do any better?

THE CASE OF SPHERE WITH UNLABELLED PUNCTURES

THEOREM

Let Γ be a pants universal family of curves on with minimal size on the sphere with n (unlabelled) punctures. Then

$$|\Gamma| = O(n^2) \quad \text{and} \quad |\Gamma| = \Omega(n \log(n)).$$

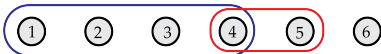
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- Upper bound: label the boundaries arbitrarily and consider standard curves encompassing consecutive punctures.



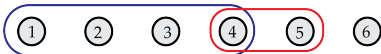
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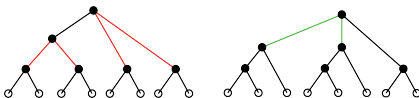
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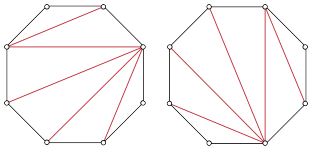
- For $i \leq \frac{n}{2}$ at least $\lfloor \frac{n}{i} \rfloor$ curves that separate i boundaries must exist in the pants universal family.

→ $\sum \lfloor \frac{n}{i} \rfloor = \Omega(n \log n)$.

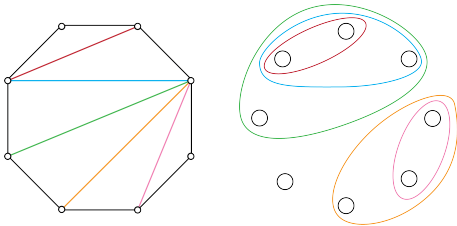
Families of edges realizing all Triangulations

REALIZING ALL TRIANGULATIONS

- Two triangulations of an n -gon are of the same type, if one can be obtained from a rotation and/or reflection of the other.



- This is similar to the case of spheres with (unlabelled) boundaries.

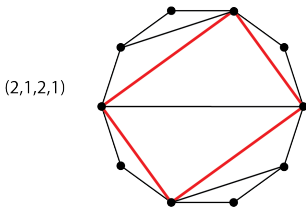


REALIZING ALL TRIANGULATIONS

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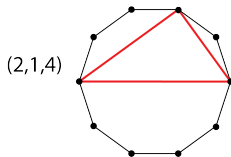
Let E be a family of edges that realize all the triangulations of an n -gon. We have $|E| = O(n^2)$ and $|E| = \Omega(n^{2-\epsilon})$ for any $\epsilon > 0$.

- Upper bound follows by computing all the diagonals in the n -gon.
- Lower bound:
 - Any such family also needs to realize all types of m -cycles.
- There exist $\Theta(n^{m-1})$ types of m -cycles.
- Any such family defines a graph with $|E|$ edges.
- any graph with $|E|$ edges has $O(|E|^{\frac{m}{2}})$ m -cycles. (Rivin)

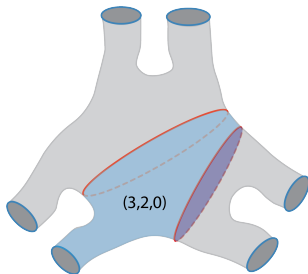


REALIZING ALL TYPES OF PANTS/ TRIANGLES

- A simpler question is to ask for the size of a family of curves that realize all types of triangles.



- or on a sphere with n (unlabelled) punctures, a family that realizes all types of pairs of pants:



SUMMARY AND CONCLUSION

	Lower bound	Upper bound
Surface of genus g without punctures	$\Omega(n^{\frac{4}{3}})$	3^{2g-1}
Sphere with n labelled punctures	$2^{n-1} - n - 1$	3^{n-1}
Sphere with n unlabelled punctures	$\Omega(n \log(n))$	$O(n^2)$

	Lower bound	Upper bound
realizing triangulations in an n -gon	$\Omega(n^{2-\epsilon})$	$O(n^2)$

	Lower bound	Upper bound
realizing all types of pants in $S_{0,n}$	$\Omega(n)$	$O(n^{\frac{4}{3}} \log^{\frac{2}{3}} n)$
realizing all triangles in an n -gon	$\Omega(n^{\frac{4}{3}})$	$O(n^{\frac{4}{3}} \log^{\frac{2}{3}} n)$

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Thank You!