Scale invariant random geometries from mating of trees

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Basics of Quantum Gravity

16 May 2023 to 16 November 2023
Europe/Zurich timezone

The Basics of Quantum Gravity online school is a series of lectures on the key concepts and techniques involved in the research of Quantum Gravity. It is dedicated to young researchers who are tackling this great challenge in theoretical physics. The goal is to bridge several paths and approaches to quantum gravity and provide young researchers with the tools to work on all of these frameworks and utilize various formalisms.

The launching lecture for this event will take place on May 16th. The event will run throughout Europe's spring and summer of 2023, featuring lectures on String Theory, Loop Gravity, the perturbative and non-perturbative renormalization flow of General Relativity, and quantum information in Gravity. Additional lectures on other topics will be planned for the fall and winter.

This is an initiative by the International Society for Quantum Gravity. It will be followed by a series of workshops on more advanced topics.
Quantum Gravity Approaches

Classical Gravity works here

quantum perturbations + very high energy
Quantum Gravity **APPROACHES**

**Classical Gravity works here**

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**Non-renormalizable**

\[
\text{Prob}(\text{?}) = ?
\]

No predictivity!

- quantum perturbations
- very high energy

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Saturn

*HST - September 12, 2021*

- WFC3/UVIS F395N
- WFC3/UVIS F502N
- WFC3/UVIS F631N

75,000 miles

120,000 kilometers
Quantum Gravity Approaches

Asymptotic Safety

+ non-perturbative!

+ very high energy
Quantum Gravity Approaches

Asymptotic Safety

- non-perturbative!
- very high energy

scale invariance = Predictive!

fixed point of RG flow
Quantum Gravity Approaches

Asymptotic Safety

Asymptotic Safety

Approaches:

- non-perturbative!
- very high energy

Dynamical Triangulations

Spacetime @ fixed point

AS review [Eichhorn, 1810.07615]
CDT review [Loll 1905.08669]
How to study discrete surfaces?

Planar map

Walk
How to study discrete surfaces?

Planar map

Walk
HOW TO STUDY DISCRETE SURFACES?

Walk Planar map

How to study discrete surfaces?
HOW TO STUDY DISCRETE SURFACES?

Walks 

Planar maps 

\[ Z_n = \frac{4^n}{(n+1)(2n+1)} \binom{3n}{n} \xrightarrow{n \to \infty} \sqrt{\frac{3}{16\pi}} 3^{3n} n^{-5/2}. \]

Kreweras '65
Bernardi 0605320
Gwyne, Holden, Sun 1905.06757

count
Walks n steps

count
Planar maps

\( n \) vertices.
Brownian Motion

Random walk

scaling limit

$n \to \infty$

Random planar map

scale invariant!

$B(t) \sim \frac{1}{a} B(a^2 t)$
Random walk

Random planar map

scaling limit

$\lim_{n \to \infty} \approx \infty$
Liouville Quantum Gravity

\[
S_L = \frac{1}{4\pi} \int d^2x \sqrt{\hat{g}} \left( \hat{g}^{ab} \partial_a \phi \partial_b \phi + Q \hat{R} \phi + 4\pi \hat{\mu} e^{\gamma \phi} \right)
\]

\[
\hat{\mu} > 0
\]

\[
Q = \frac{\gamma}{2} + \frac{2}{\gamma}
\]

\[
\gamma \in (0, 2)
\]

\[
g_{ab} = e^{\gamma \phi} \hat{g}_{ab}
\]

Random walk

Random planar map

Path integral

\[
e^{-S_L}
\]
Random walk

Brownian Motion

scaling limit
\[ n \to \infty \]

Random planar map

Random Surface

scaling limit
\[ n \to \infty \]
Brownian Motion

Random walk

Random graph

Random Surface

scale invariant?

\[ Z_n \xrightarrow{n \to \infty} C K^n n^{\gamma - 2} \]
The answer: **Mating of Trees**

\[ \text{Cov}(X, Y) = -\cos\left(\frac{\pi}{4} \gamma^2\right) \]
The answer: **MATING OF TREES**

\[ \gamma_s = 1 - 4/\gamma^2 \]

**String susceptibility**

2D Brownian Motion

\[ \text{Cov}(X, Y) = -\cos\left(\frac{\pi}{4}\gamma^2\right) \]

Scale Invariant Random Surface

\[ G_{ab} = \epsilon_{ab} \hat{G}_{ab} \]

Duplantier, Miller, Sheffield [1409.7055]
GENERALIZATION

3D Brownian Motion

\[
\begin{pmatrix}
1 & -\cos(\alpha) & -\cos(\gamma) \\
-\cos(\alpha) & 1 & -\cos(\beta) \\
-\cos(\gamma) & -\cos(\beta) & 1
\end{pmatrix}
\]
GENERALIZATION

3D Brownian Motion

Scale Invariant Random Geometry
How to simulate Brownian excursions

Probability density: standard BM started at $x$ remains in wedge after time $t$

$$P_t^C(x, y)$$

Angular

$$\begin{aligned}
L_{S^{d-1}} m_i(\tilde{x}) &= -\lambda_i m_i(\tilde{x}) \\
m_i(\tilde{x}) &= 0
\end{aligned}$$

for $\tilde{x} \in W \cap S^{d-1}$, for $\tilde{x} \in \partial W \cap S^{d-1}$. 

Correlated

Uncorrelated
How to simulate Brownian excursions

Probability density: standard BM started at \( x \) remains in wedge after time \( t \)

\[
P_t^C(x, y) = ct^{\gamma_s-2} + O(t^{\gamma_s-2})
\]

\[
\gamma_s = 1 - \sqrt{\lambda_1 + \left(\frac{d}{2} - 1\right)^2}
\]

"String susceptibility"
CRITICAL EXPONENTS: STRING SUSCEPTIBILITY

\[ \mathbf{C} = \begin{pmatrix} 1 & -\cos(\alpha) & -\cos(\gamma) \\ -\cos(\alpha) & 1 & -\cos(\beta) \\ -\cos(\gamma) & -\cos(\beta) & 1 \end{pmatrix} \]

\[ \alpha = \frac{\pi}{4} \gamma^2 \]
HAUSDORFF DIMENSION

volume / length $\rightarrow \text{Vol} \sim l^{dh}$

e.g. $\text{Area} \sim r^2$

# of vertices $\sim (\text{graph \ distance})^{dh}$

2D building blocks
HAUSDORFF DIMENSION

\[ \lim_{n \to \infty} n^{1/d_H} \rho_n(n^{1/d_H} x) = \rho(x) \]
HAUSDORFF DIMENSION

$$\lim_{n \to \infty} n^{1/d_H} \rho_n(n^{1/d_H} x) = \rho(x)$$

$$\left( \frac{n}{n_0} \right)^{-1/d_H} \left( a + b \left( \frac{n}{n_0} \right)^{-\delta} \right)$$
HAUSDORFF DIMENSION MEASUREMENTS

\[ d_{DG} = \frac{\gamma^2}{2} + \frac{\gamma}{\sqrt{6}} + 2 \]

\[ d_{W} = \frac{\gamma^2}{4} + \sqrt{\left(\frac{\gamma^2}{4} + 1\right)^2 + \gamma^2 + 1} \]

\[ g_{ab} = e^{\gamma \phi} \hat{g}_{ab} \]

\[ Cov(X, Y) = -\cos\left(\frac{\pi}{4} \gamma^2\right) \]
HAUSDORFF DIMENSION MEASUREMENTS

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Outlook

Math

- New universality classes of higher dimensional scale-invariant random geometries
- more evidence for Hausdorff dimension predictions
- Topology?
- Random walk for special points

Physics

- Asymptotic Safety fixed point?
- matching critical exponents
  - spectral dimension measurements
  - Hausdorff dimension in AS
- Field theory of scaling limit?
  - analogous to LQG