

Greedy Tamari intervals



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Des tamaris gloutons ?



Des tamaris



Tamarinier, feuilles et fruits

Des tamaris gloutons ?



Des tamaris



Tamarinier, feuilles et fruits



Un tamarin empereur

Des tamaris gloutons ?



Des tamaris



Tamarinier, feuilles et fruits



Un tamarin empereur



Un glouton

Des tamaris gloutons ?



Des tamaris



Dov Tamari (1911-2006)



Un tamarin empereur



inier, feuilles et fruits

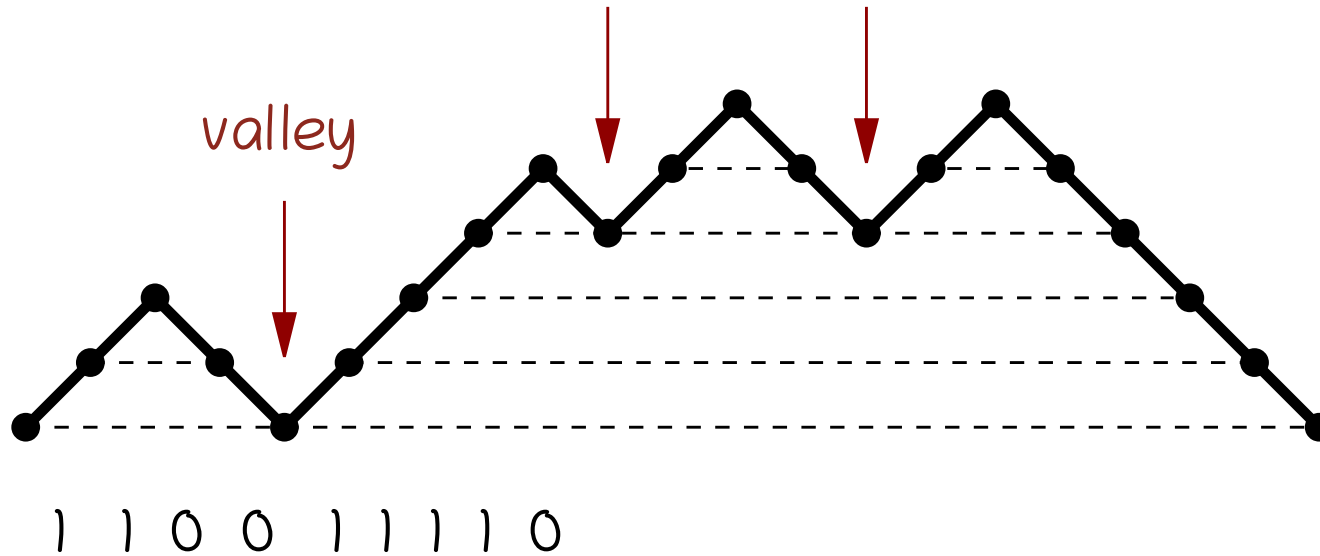


Un glouton

I. Tamari orders on Dyck paths

Dyck paths

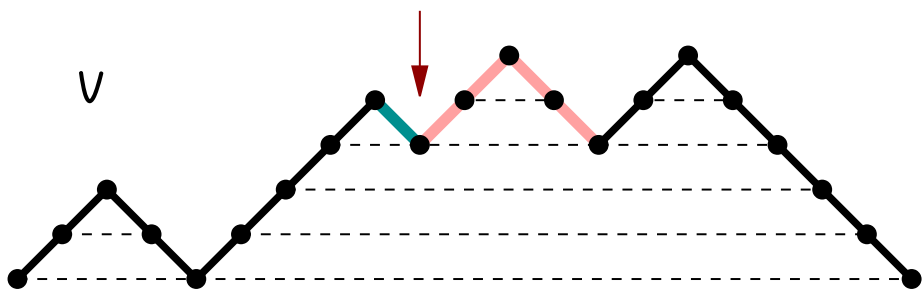
A Dyck path of **size** $n=10$ (size=number of up steps)



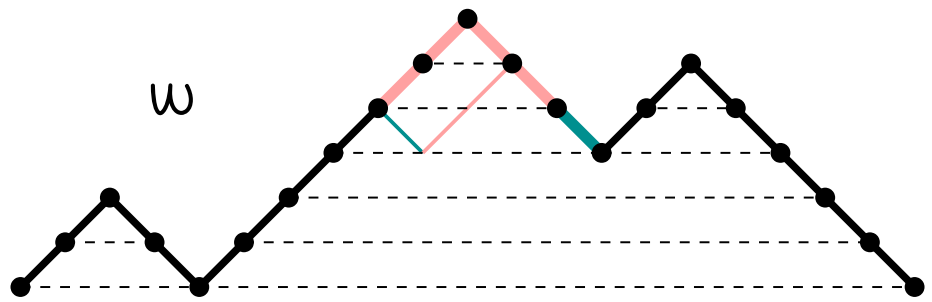
Encoding: 1 for up steps, 0 for down steps

Covering relations in ordinary/greedy Tamari orders

- **Ordinary:** swap a down step and the **shortest** Dyck path that follows



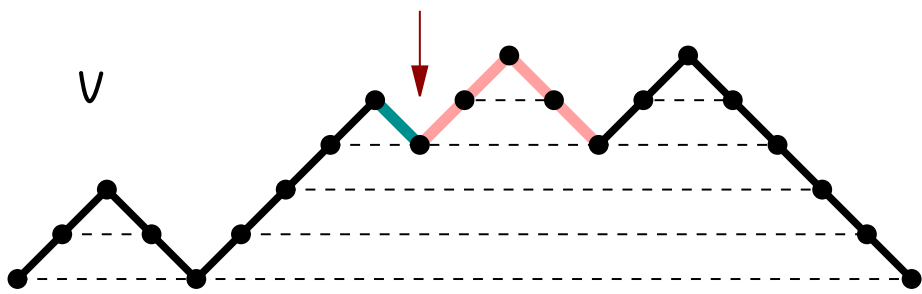
\triangleleft



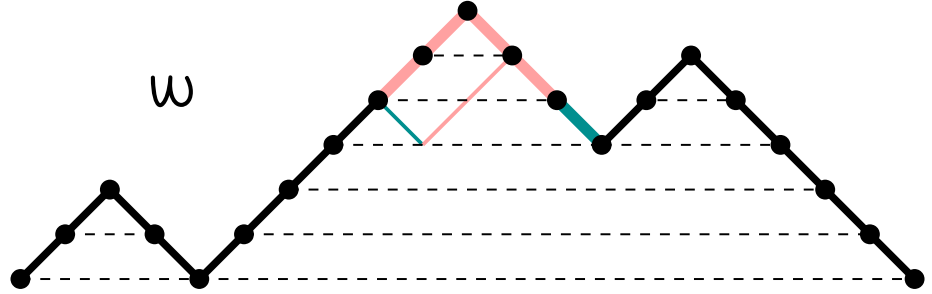
[Tamari 51]

Covering relations in ordinary/greedy Tamari orders

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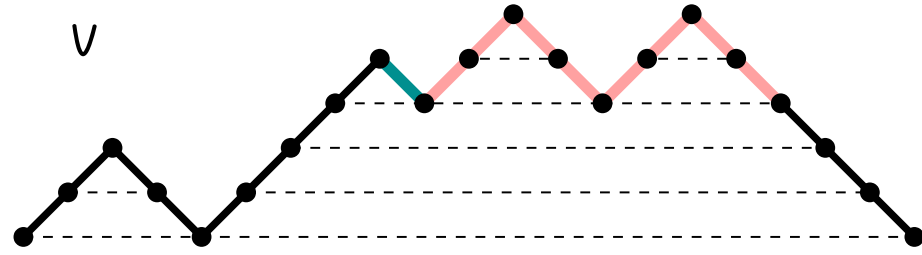


Δ

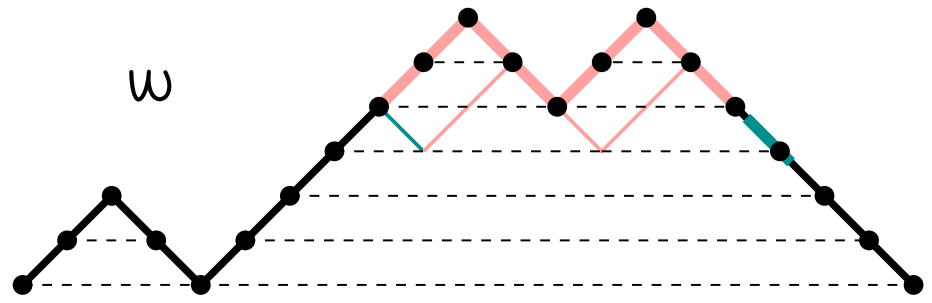


[Tamari 51]

- **Greedy:** swap a down step and the **longest** Dyck path that follows

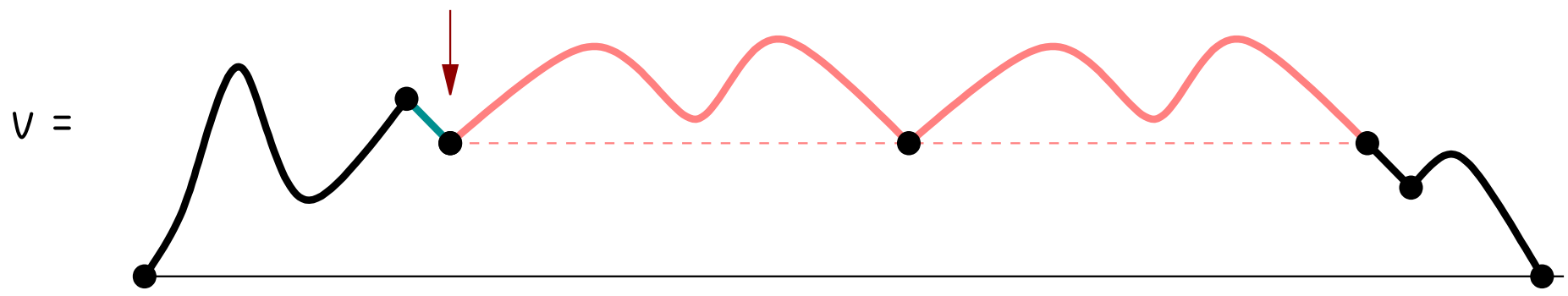


Δ

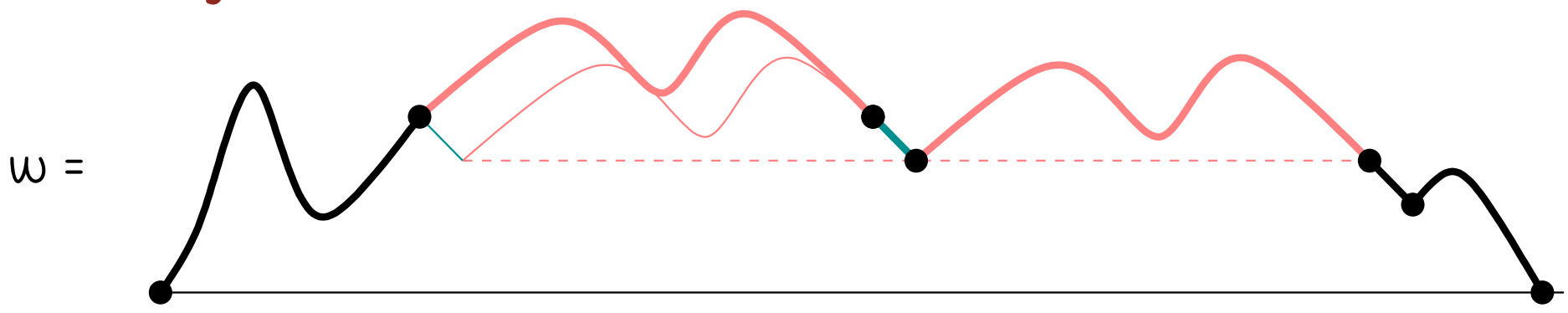


[Dermenjian 22(a)]

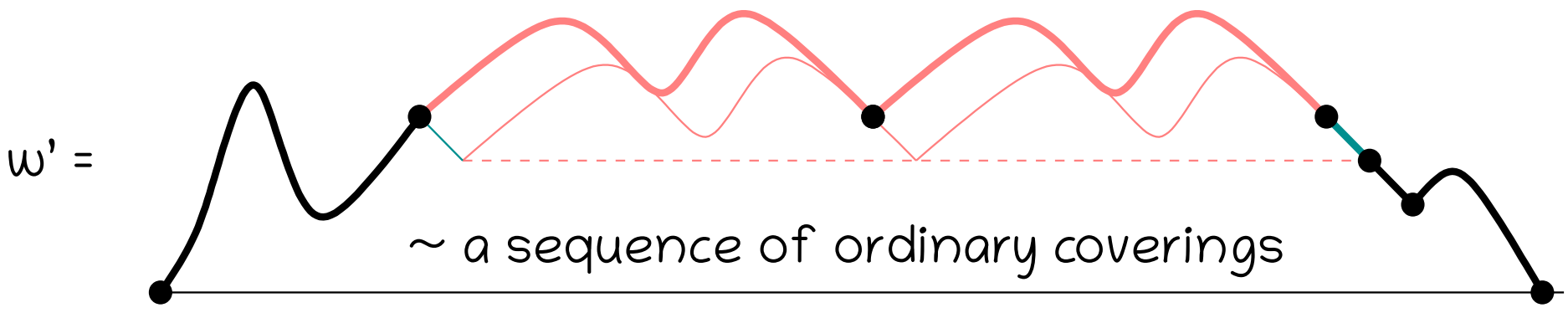
Covering relations in ordinary/greedy Tamari orders



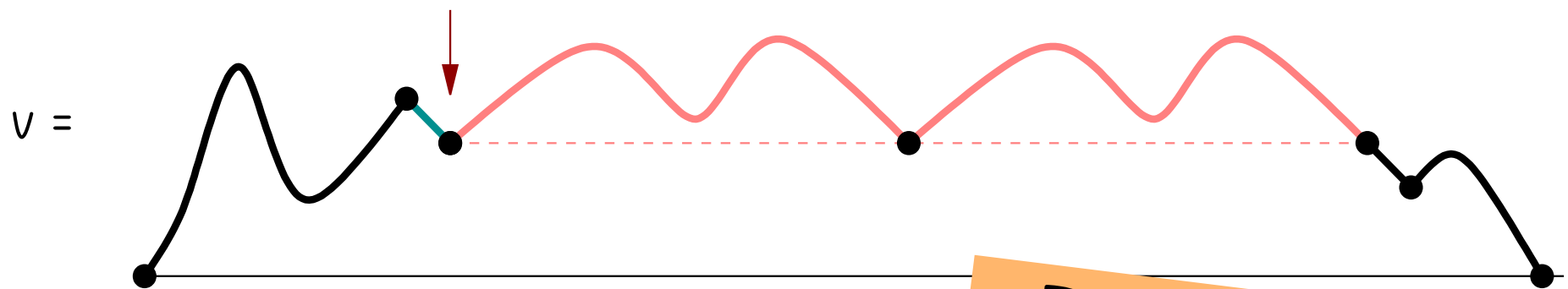
• Ordinary



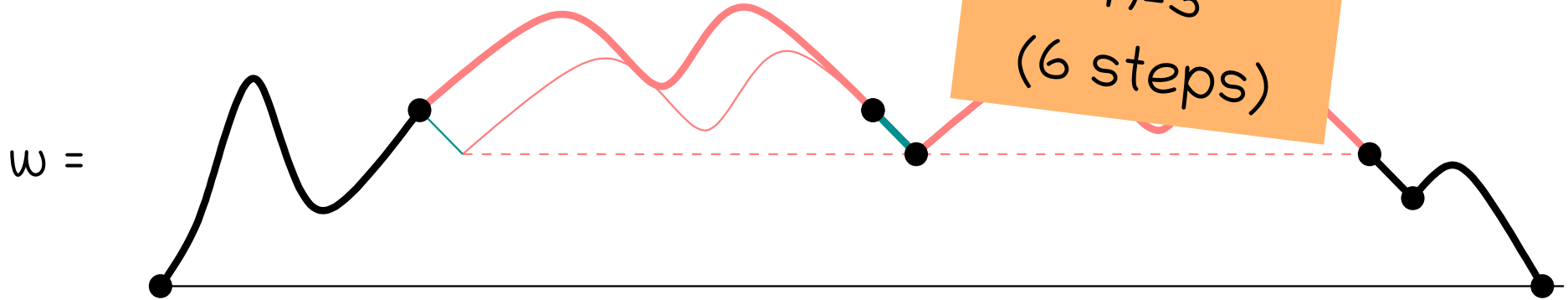
• Greedy



Covering relations in ordinary/greedy Tamari orders

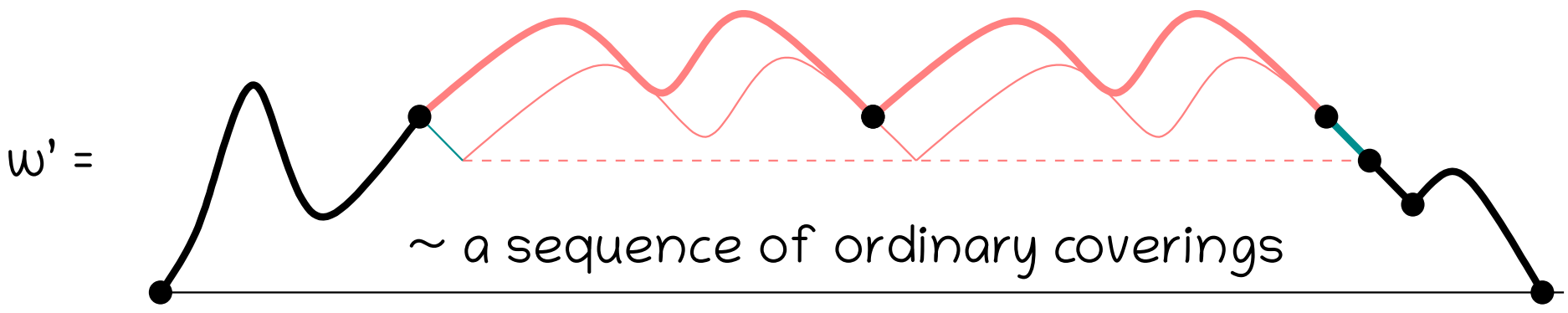


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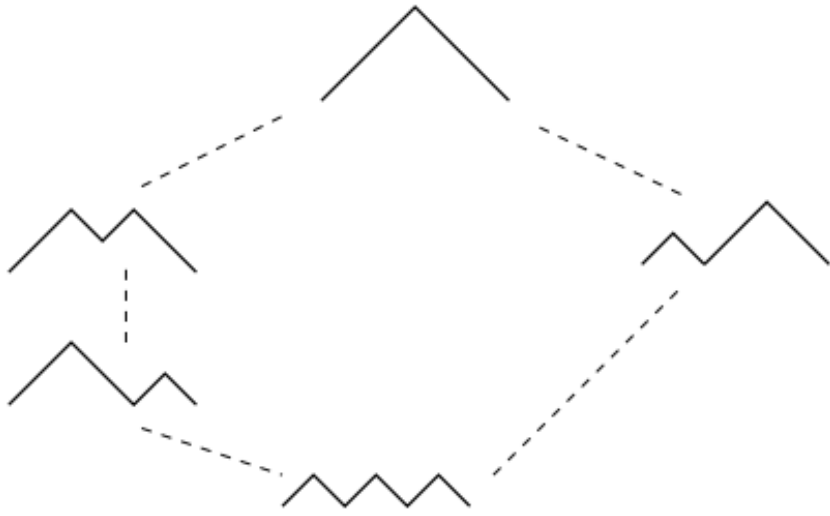
Exercise
 $n=3$
(6 steps)

• Greedy

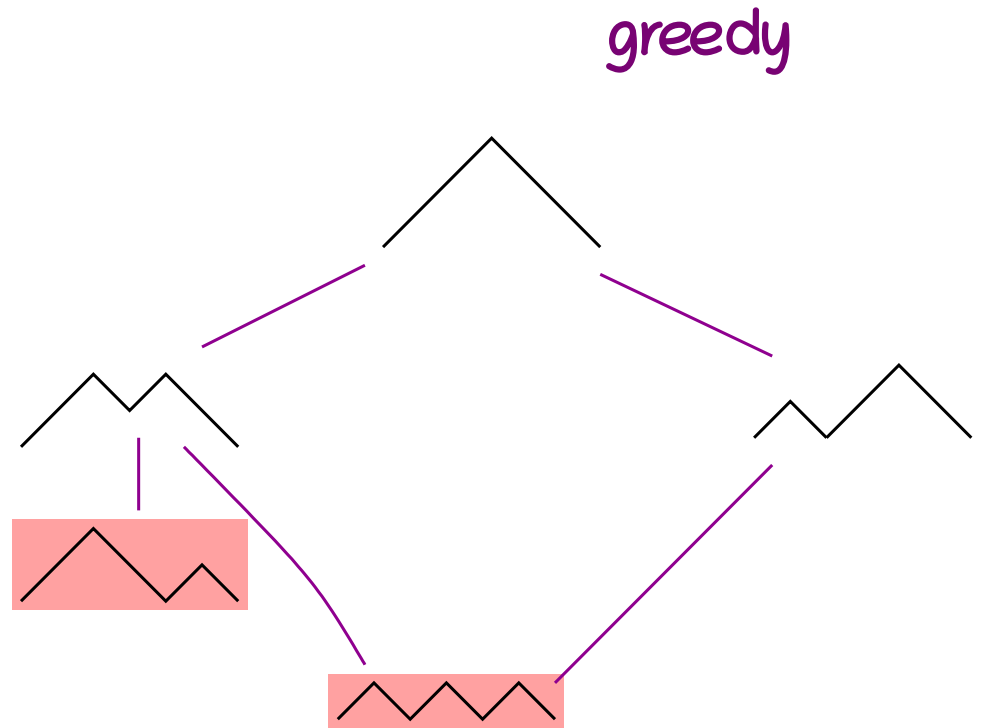


~ a sequence of ordinary coverings

The ordinary/greedy Tamari orders ($n=3$)

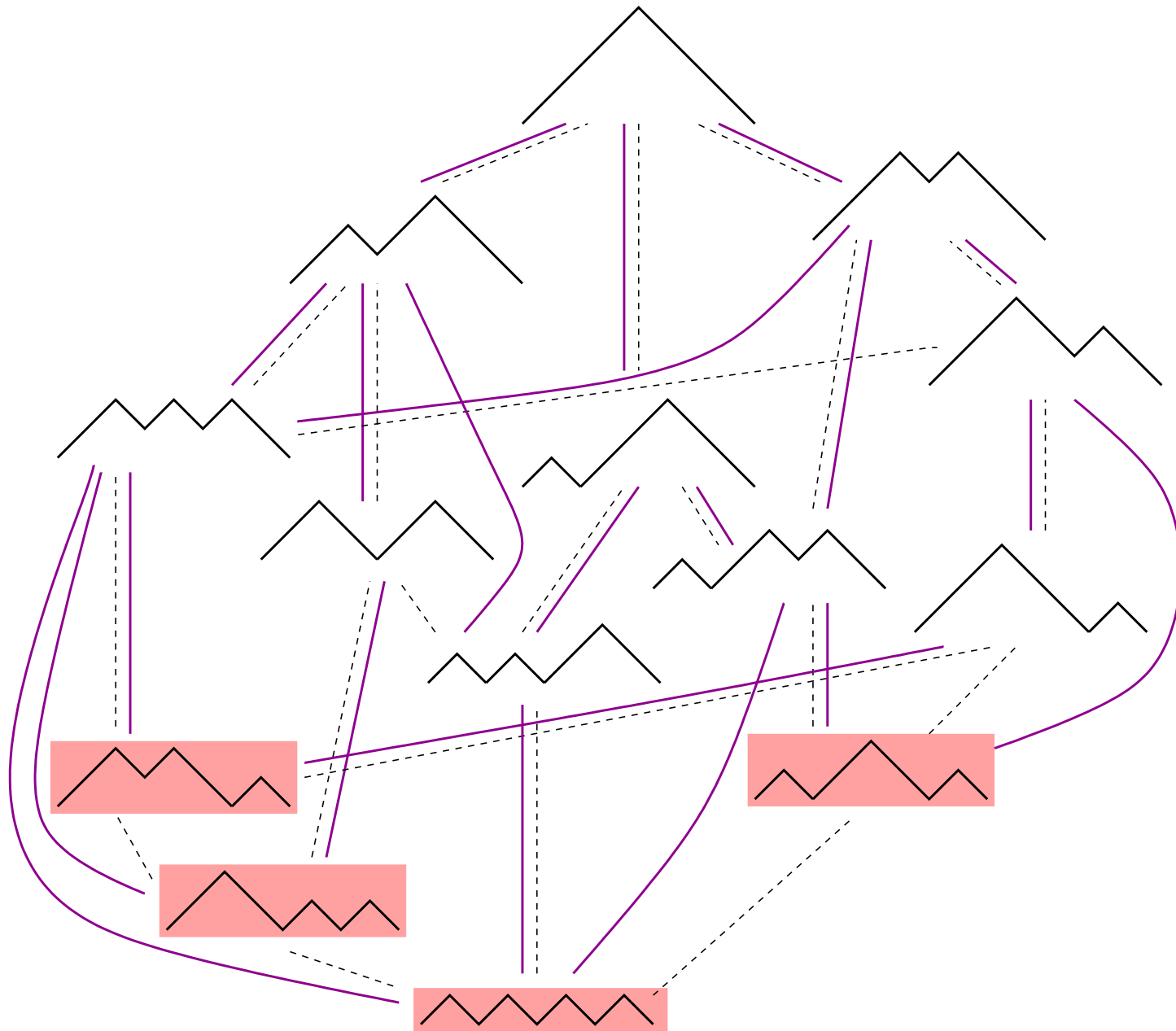


ordinary



greedy

The ordinary/greedy Tamari orders (n=4)



ordinary

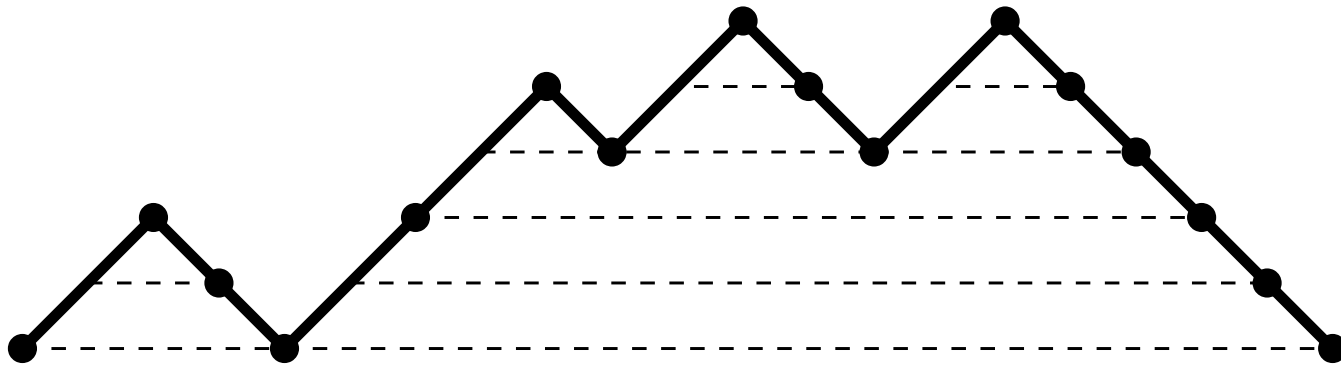
greedy

An extension to m-Dyck paths

Def. Let $m \geq 1$. An m -Dyck path is a Dyck path in which all ascent lengths are multiples of m . Equivalently, it consists of steps of size $+m$ and -1 .

The size is the number of (large) up steps.

A 2-Dyck path of size $n=5$.

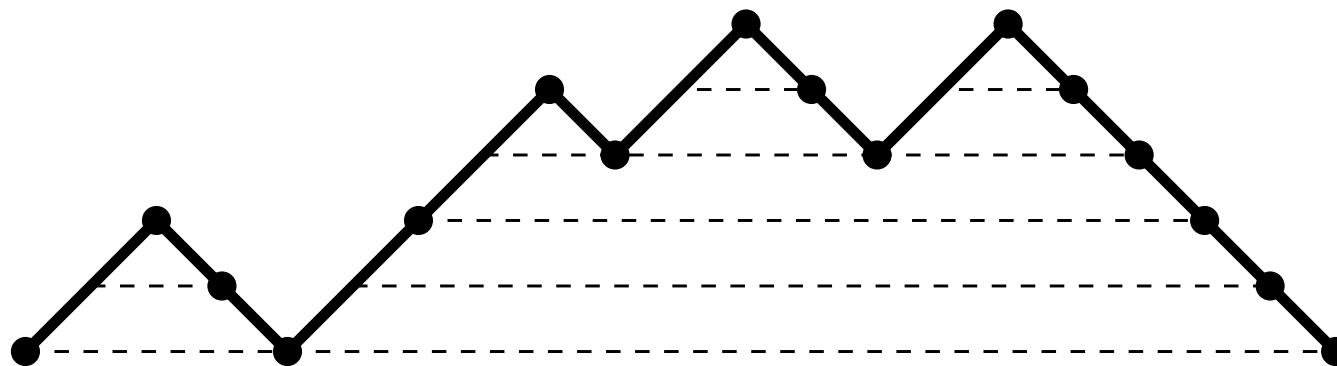


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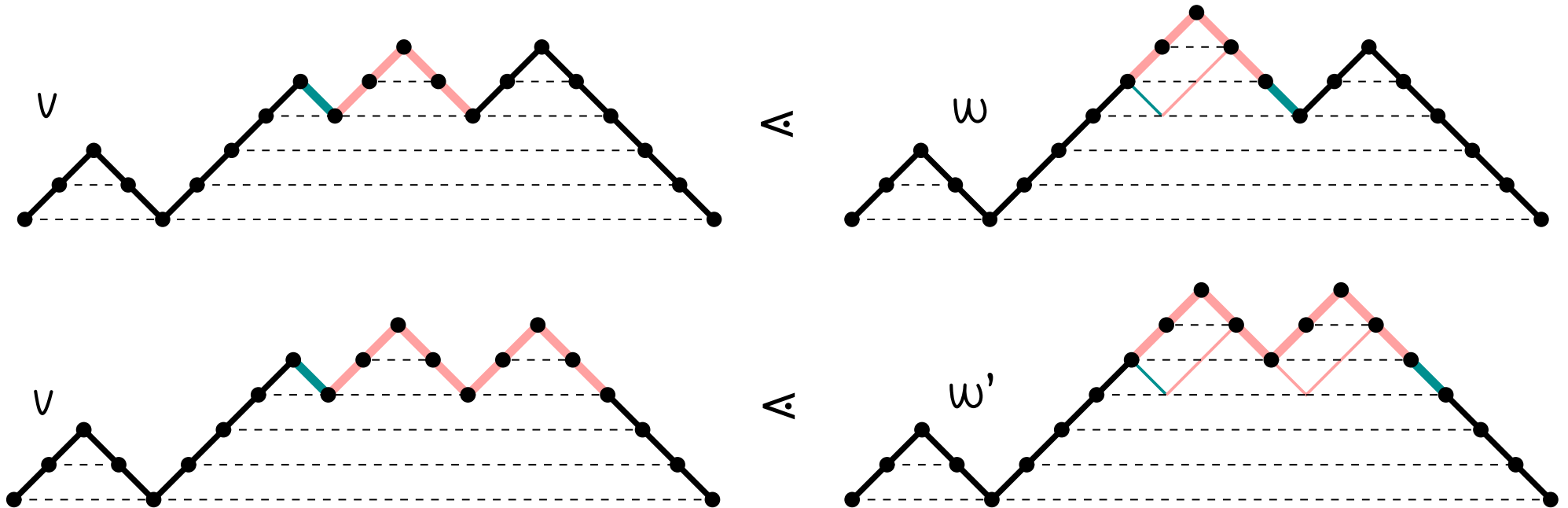
We define similarly the ordinary and greedy Tamari orders on m -Dyck paths.

[Bergeron & Préville-Ratelle 12, Dermenjian 22(a)]

m-Tamari posets as subsets of 1-Tamari

Observation: if a 1-Dyck path v has ascent lengths that are multiples of m , the same holds for any $w \succcurlyeq v$, for both orders.

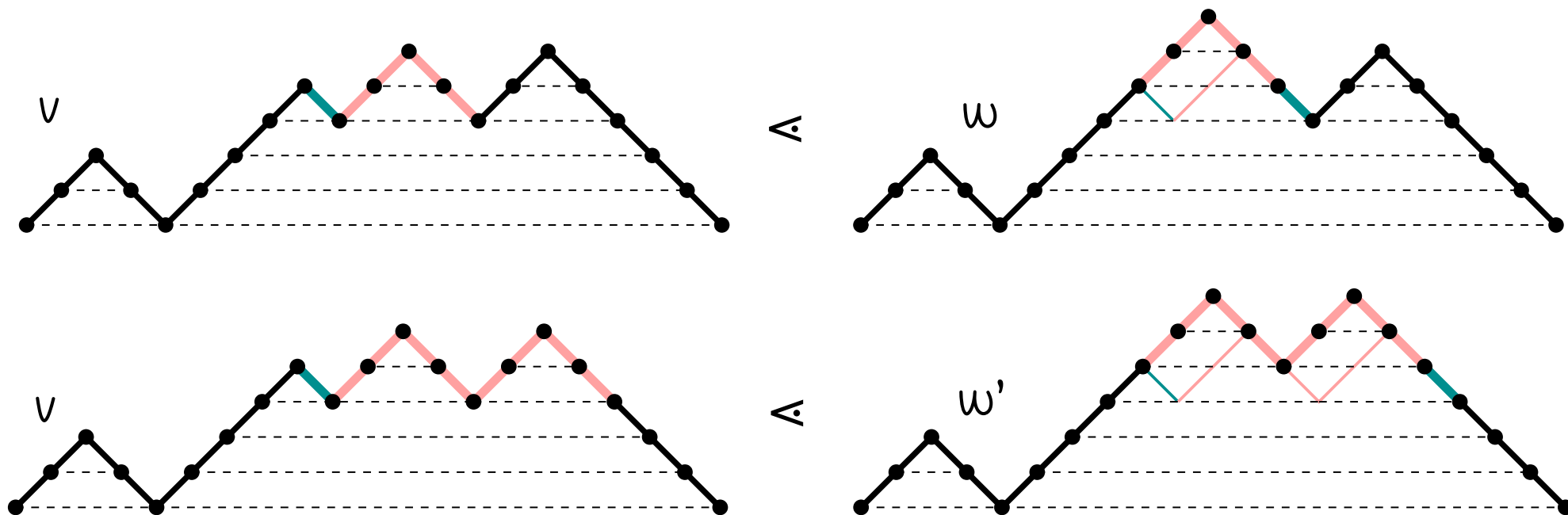
Example: $m=2$



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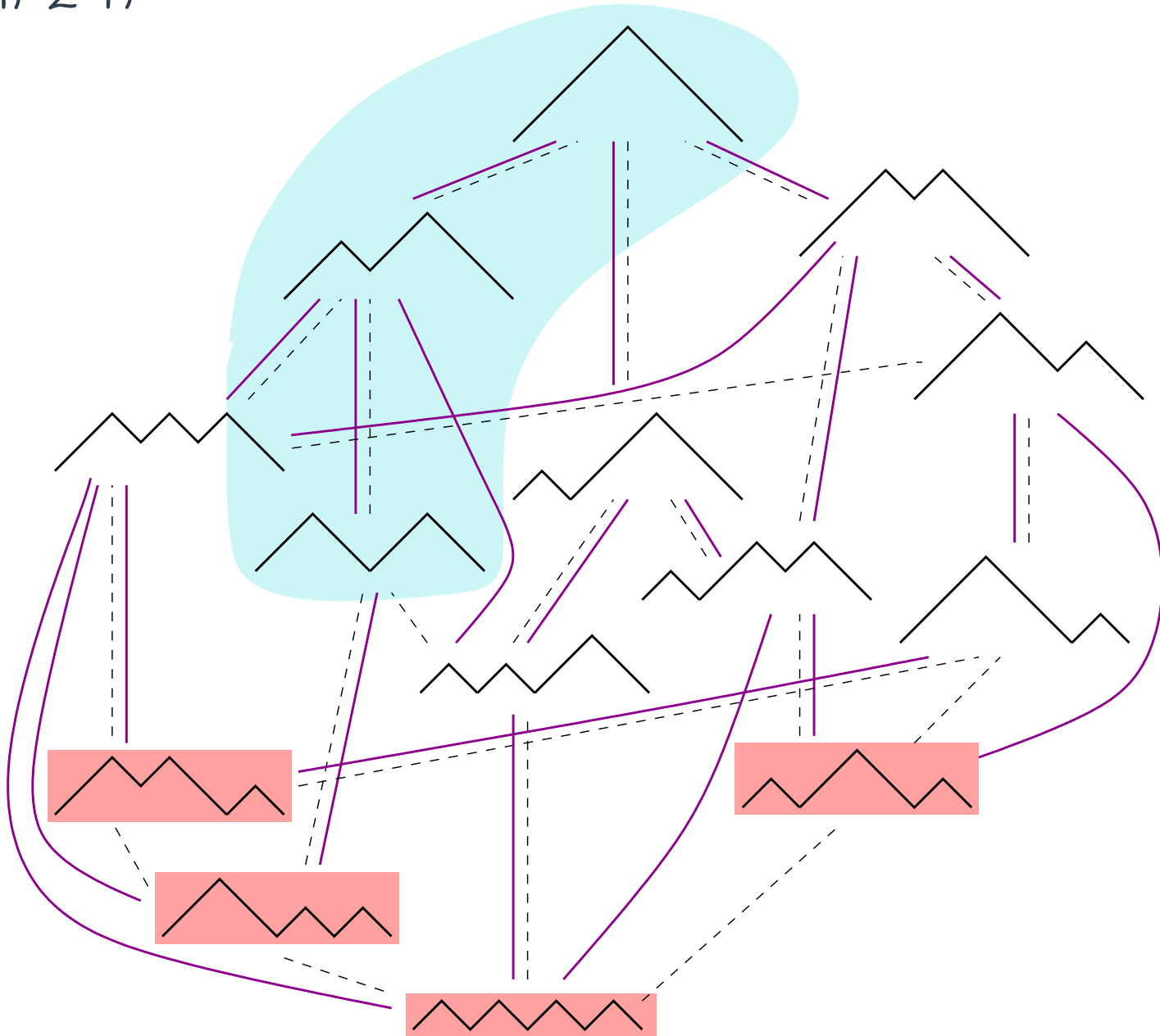
Example: $m=2$



The m -Tamari poset of size n is a subposet of the 1-Tamari poset of size mn .

m-Tamari posets as subsets of l-Tamari

Example: $m=2=n$



II. Intervals in Tamari orders

$$[v, \omega], v \preceq \omega$$

The number of ordinary m-Tamari intervals

[Chapoton 06 (m=1) -- MBM, Fusy & Préville-Ratelle 14]

The number of intervals $[v,w]$ in the m-Tamari lattice of size n (i.e., n large up steps) is:

$$t_{m,n} = \frac{m+1}{n(mn+1)} \binom{(m+1)^2n+m}{n-1}.$$

Proof: a recursive construction of intervals, involving a “catalytic” parameter: the number of **contacts** of the lower path v with the x-axis.

The number of ordinary m-Tamari intervals

[Chapoton 06 (m=1) -- MBM, Fusy & Préville-Ratelle 14]

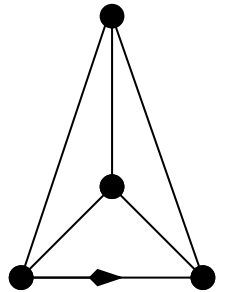
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Bijjective proof (m=1) [Bernardi & Bonichon 07]

Bijection with triangulations (no loop nor multiple edge) with n+3 vertices

$$t_{1,n} = \frac{2}{n(n+1)} \binom{4n+1}{n-1}.$$



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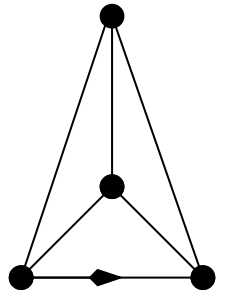
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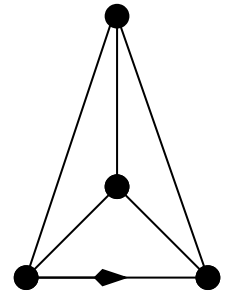
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For $m > 1$, which type of maps do these numbers count?

NEW: The number of **greedy** m-Tamari intervals

[MBM & Chapoton 23(a)]

The number of intervals $[v,w]$ in the greedy m-Tamari order of size n (i.e., n large up steps) is:

$$g_{m,n} = \frac{(m+2)(m+1)^{n-1}}{(mn+1)(mn+2)} \binom{(m+1)n}{n}.$$

Proof: a recursive construction of intervals, involving a new “catalytic” parameter: the **final descent** of the upper path w .

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Bijjective proof? We know what these numbers count!

Planar constellations

[MBM & Schaeffer 00]

The number of planar $(m+1)$ -constellations with n black faces is:

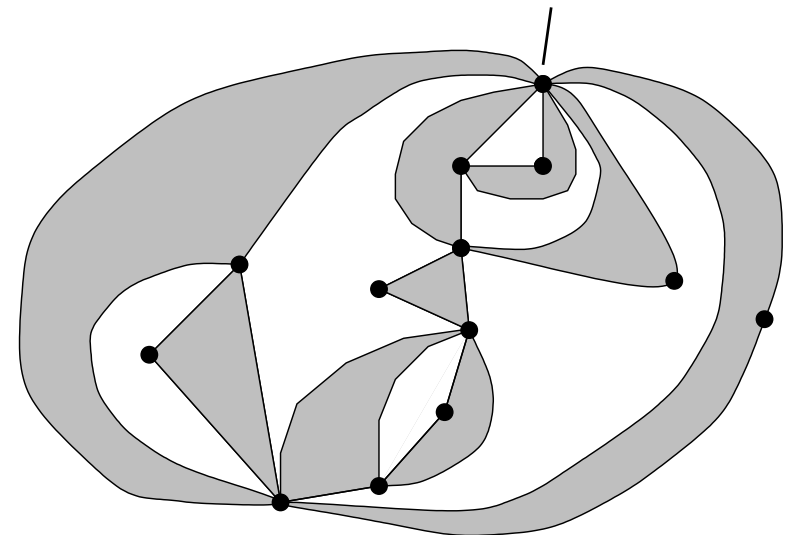
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Constellation: a rooted planar map with bicolored faces

- black faces of degree $(m+1)$
- white faces of degree $k(m+1)$, for some $k > 0$.

[Lando & Zvonkine]

$m+1=3$



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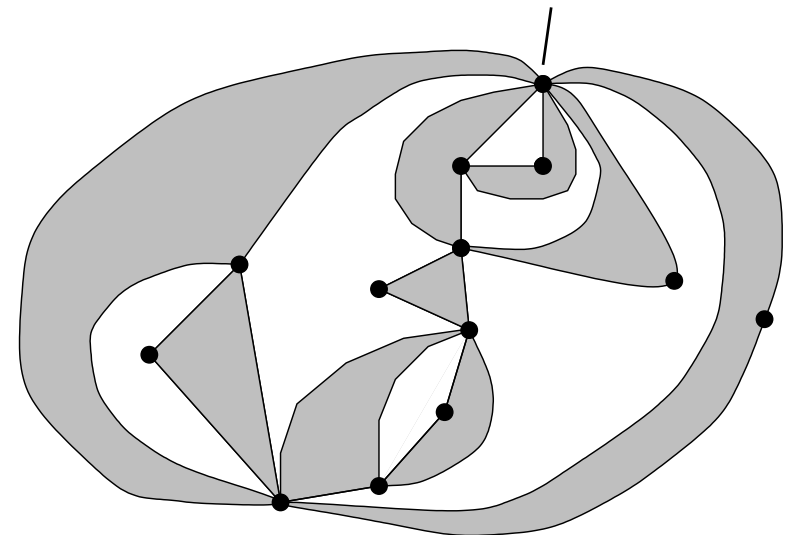
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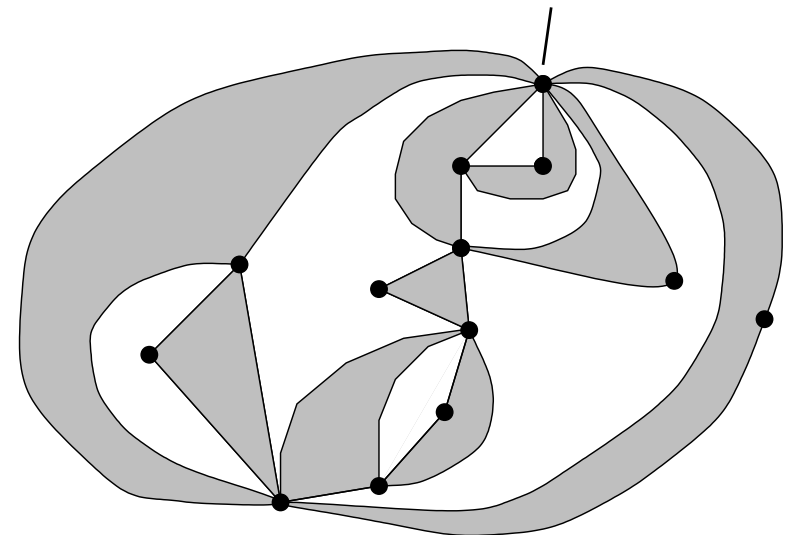
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Bijection with
greedy intervals?

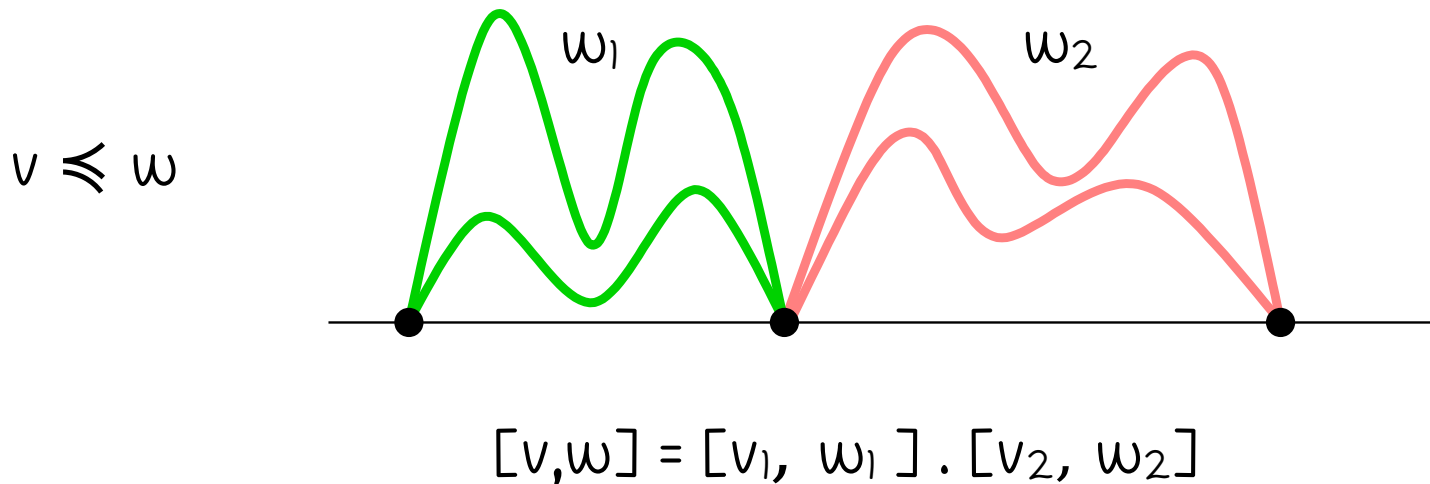
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III. A proof for greedy intervals ($m=1$)

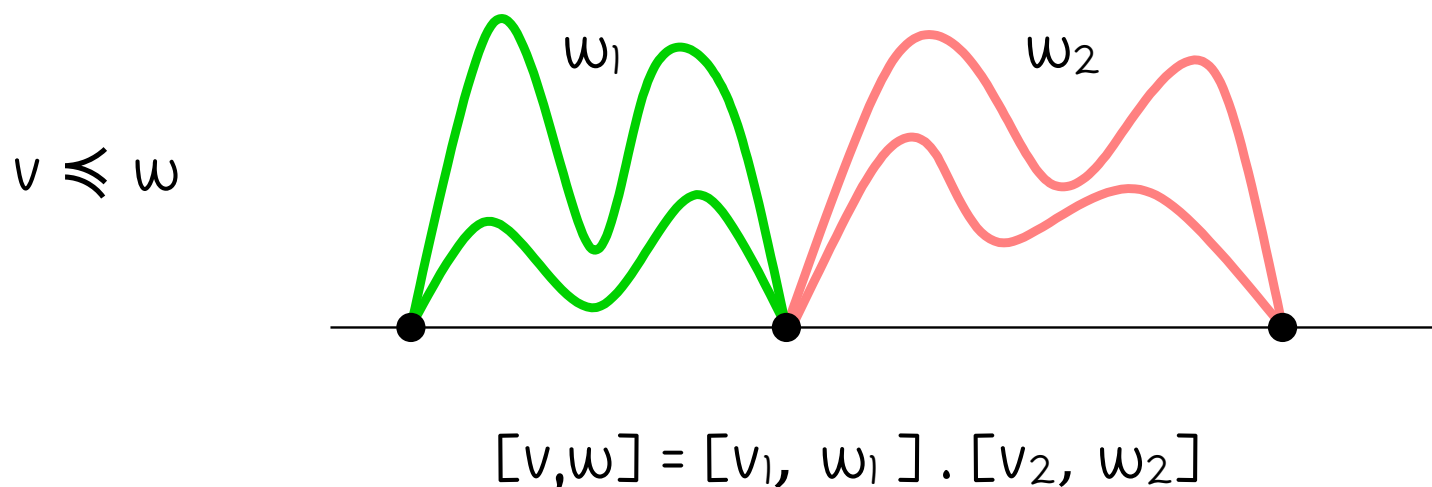
Some observations

Ordinary order and concatenation: if w_1 and w_2 are Dyck paths, and $w = w_1 \cdot w_2$, then $v \preceq w$ iff $v = v_1 \cdot v_2$ with $v_1 \preceq w_1$ and $v_2 \preceq w_2$.

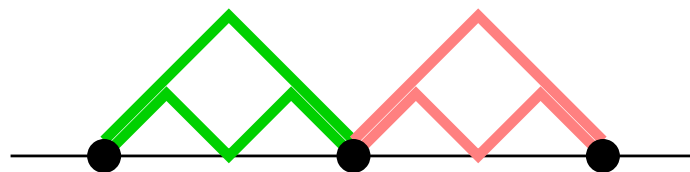


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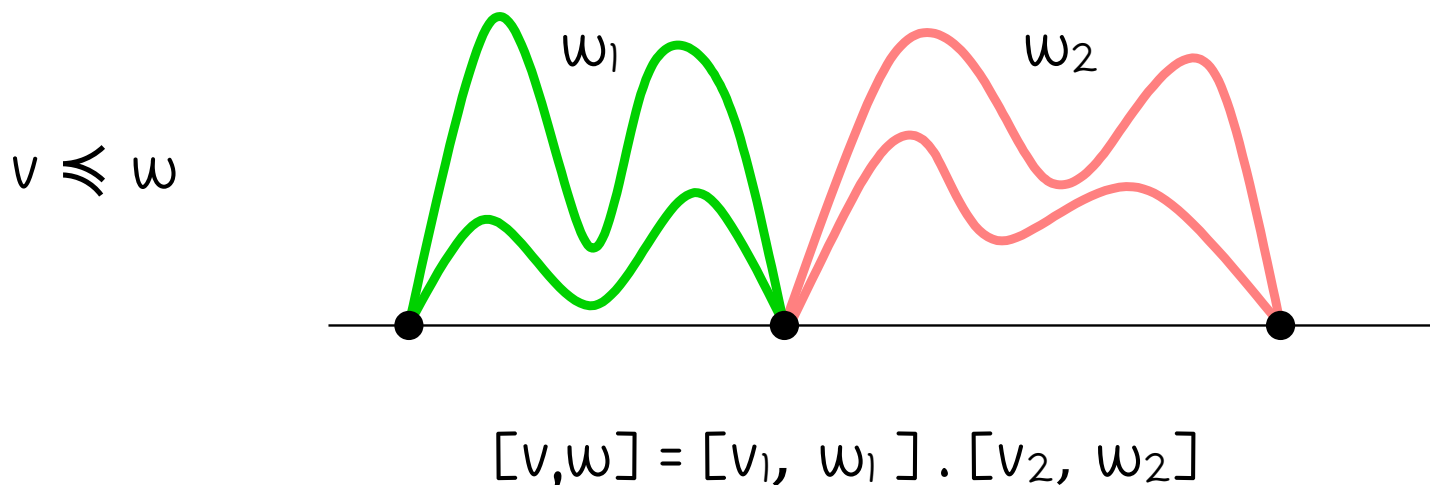
This is no longer true for greedy intervals:



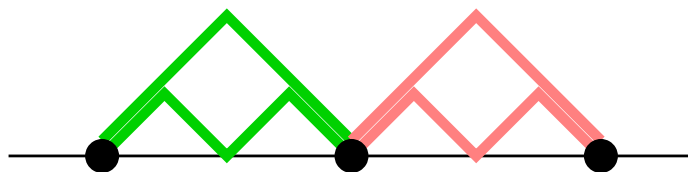
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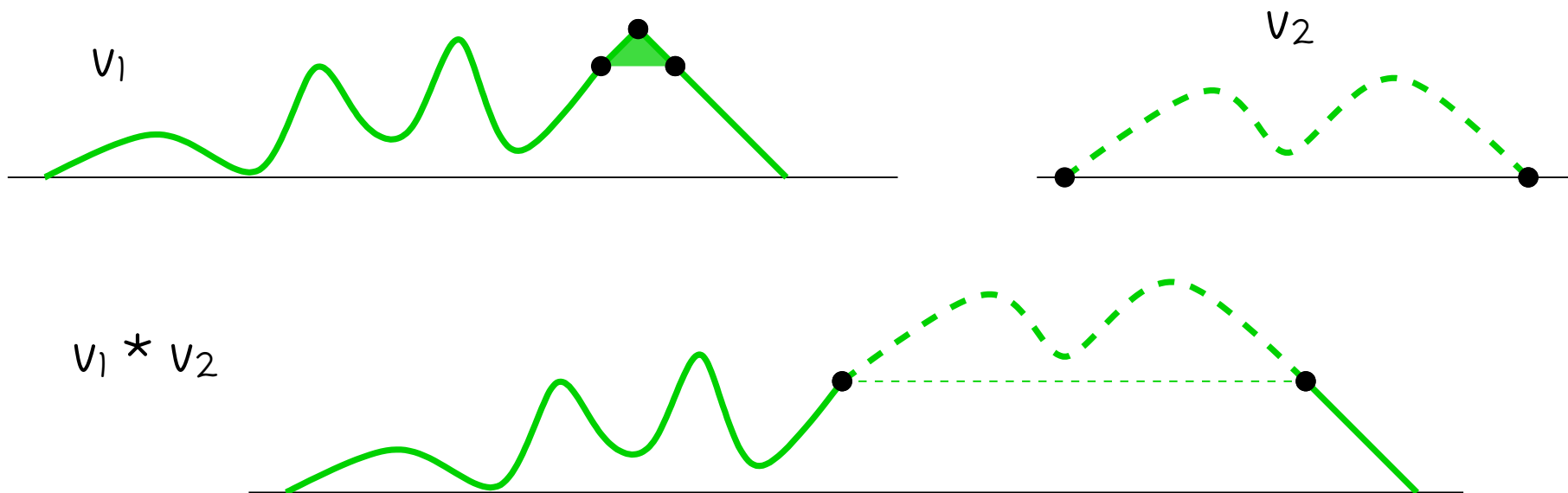
is NOT a greedy interval.

We took a different approach...
(which also works for ordinary intervals)

A new product for the greedy order

The star product

For v_1 and v_2 non-empty, let $v = v_1 * v_2$ be obtained by replacing the last peak of v_1 by v_2 .



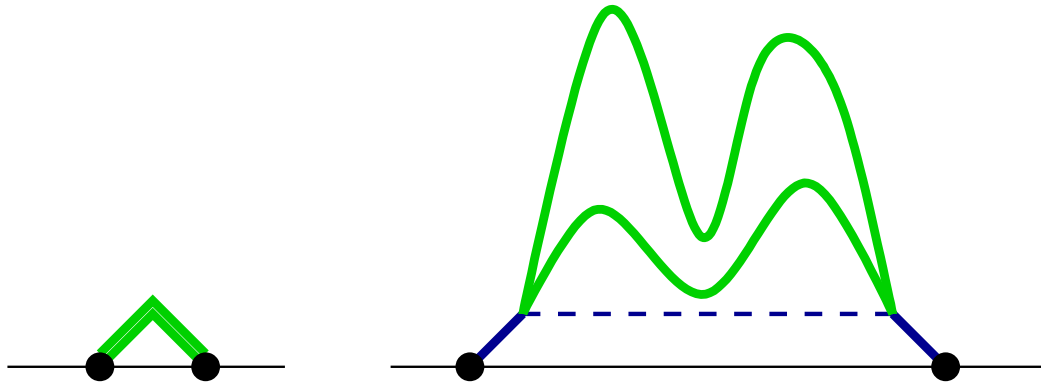
Lemma

If $v = v_1 * v_2$, then $v \preceq w$ iff $w = w_1 * w_2$ with $v_1 \preceq w_1$ and $v_2 \preceq w_2$.

Recursive description of greedy intervals ($m=1$)

Let \mathcal{J} be the set of greedy intervals $[v,w]$ of size >0 .

- **Case 1:** v is primitive (no contact apart from the endpoints)



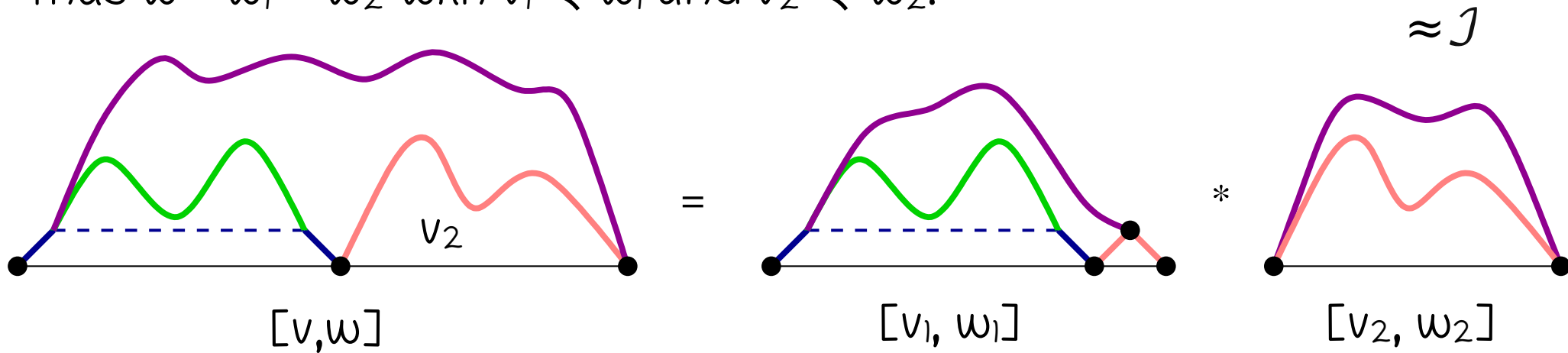
$$\mathcal{J}_1 \approx \{[10,10]\} \oplus \mathcal{J}$$

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- **Case 2:** v is **not primitive**. Then $v = v_1 * v_2$, with $v_1 = \lfloor v \rfloor$.

Thus $w = w_1 * w_2$ with $v_1 \leq w_1$ and $v_2 \leq w_2$.

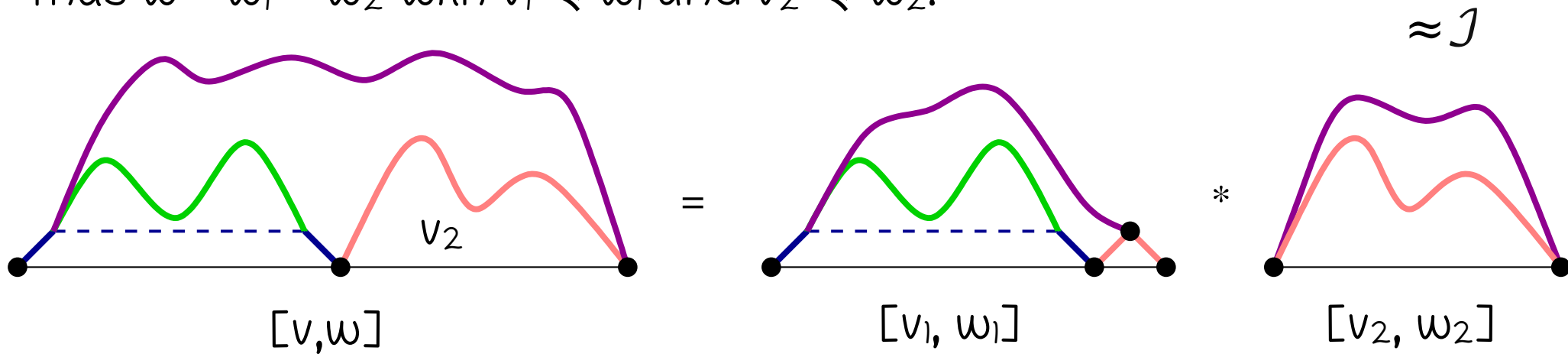


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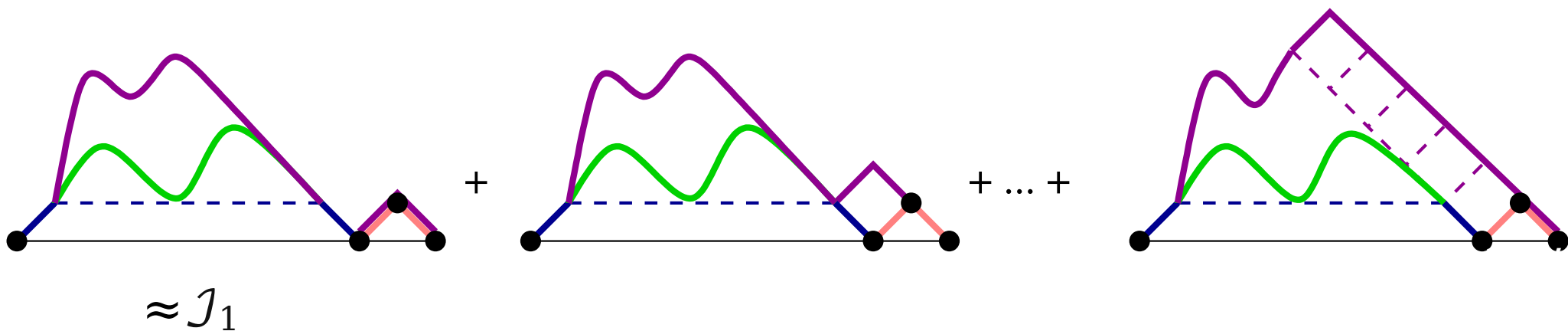
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The interval $[v_1, w_1]$:



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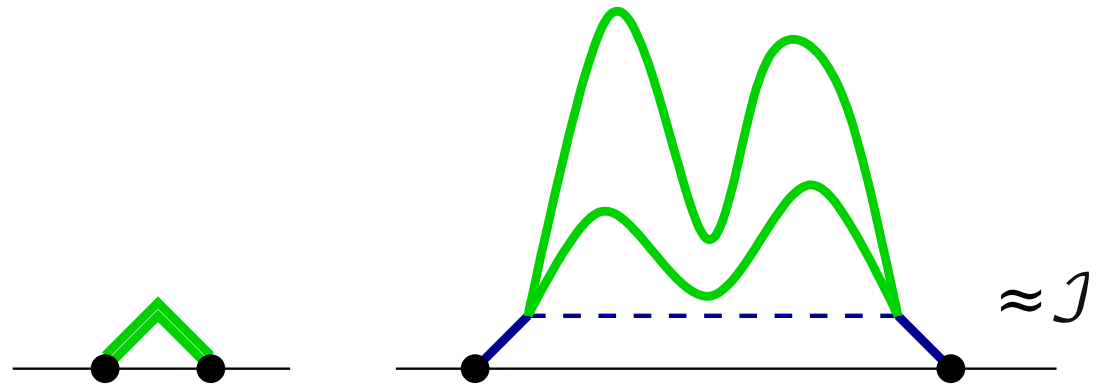
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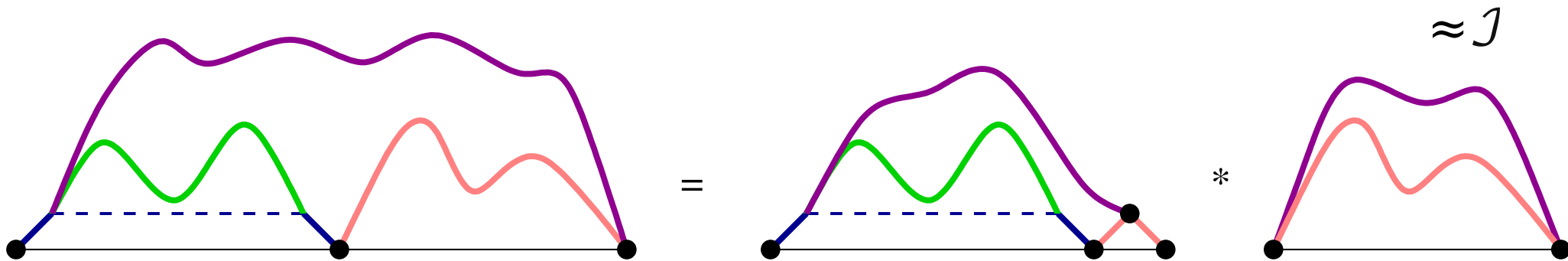
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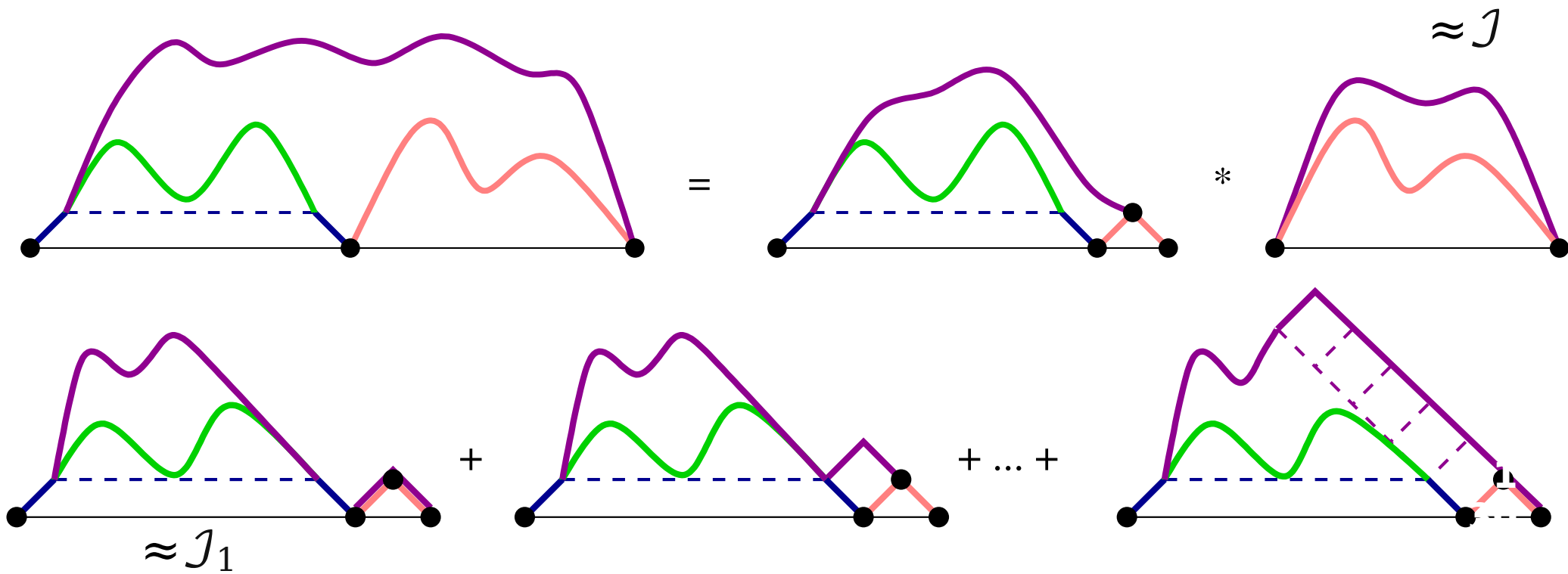
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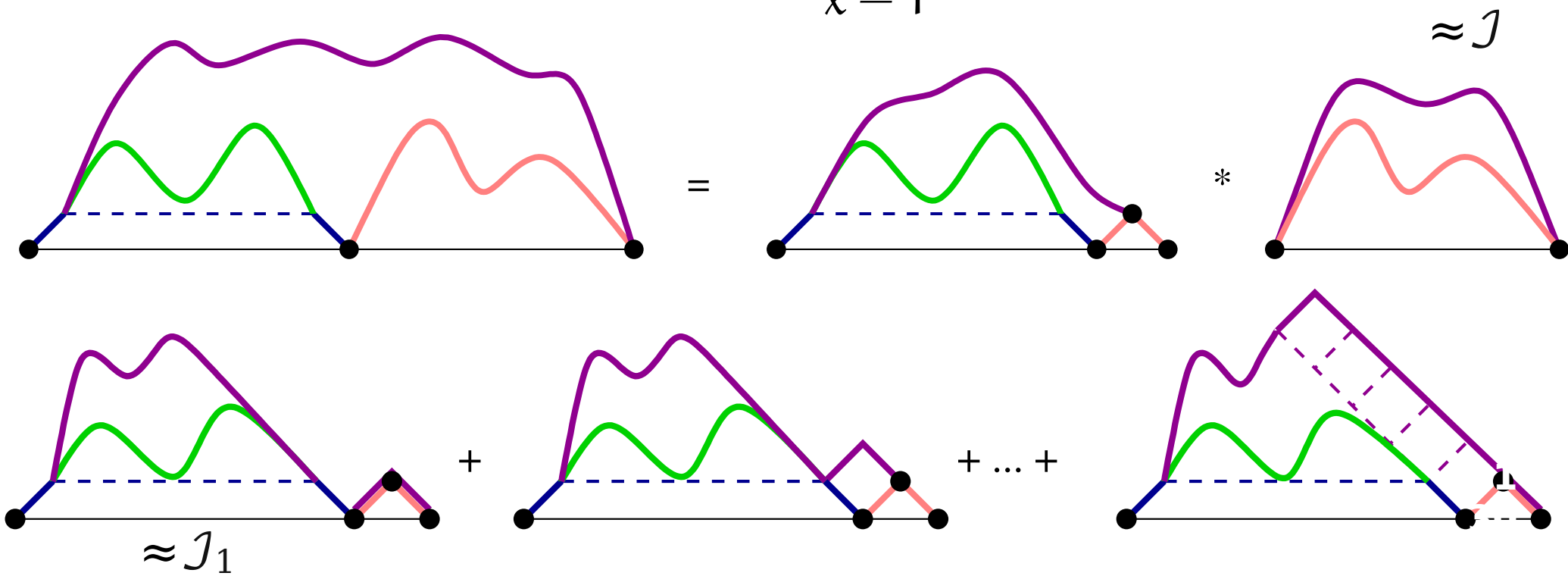
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A discrete differential equation:

$$J(x) = tx + t(1 + 2x)J(x) + t\frac{x^2J(x) - J(1)}{x-1}J(x).$$

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→ Solution via the **quadratic method** [Brown 65]

$$J(1) = \frac{(1 - 8t)^{3/2} - 1 + 12t - 24t^2}{32t^2} = \sum_{n \geq 1} \frac{3 \cdot 2^{n-1}}{(n+1)(n+2)} \binom{2n}{n} t^n.$$

**IV. When m is general:
sketch of the proof**

1/ A discrete differential equation

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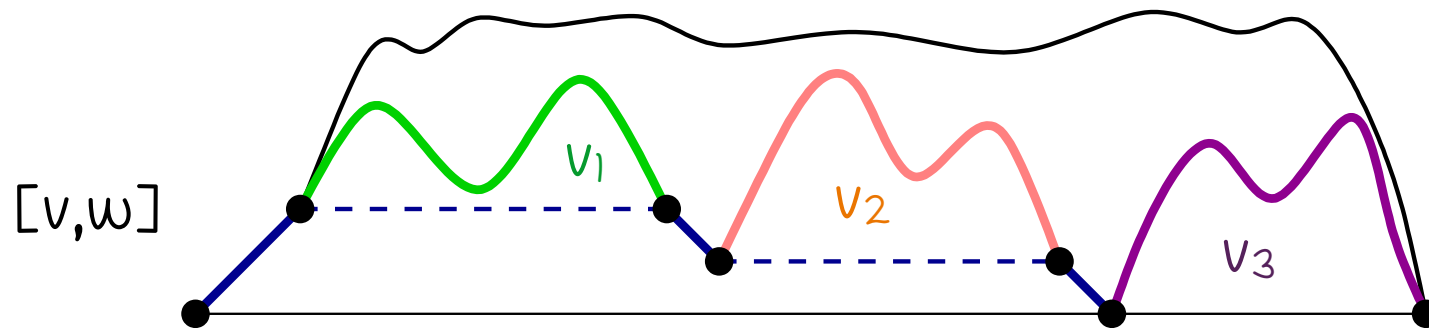
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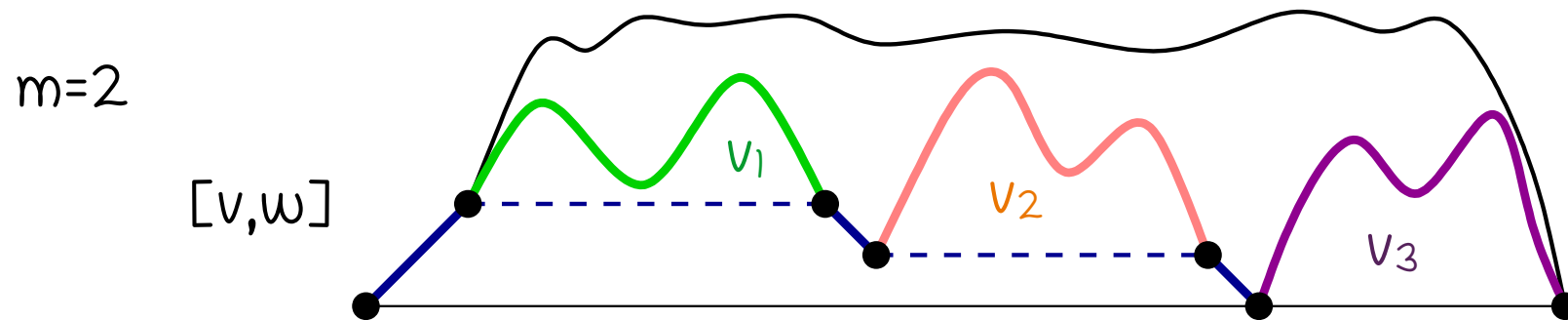
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- Their Gf's satisfy,

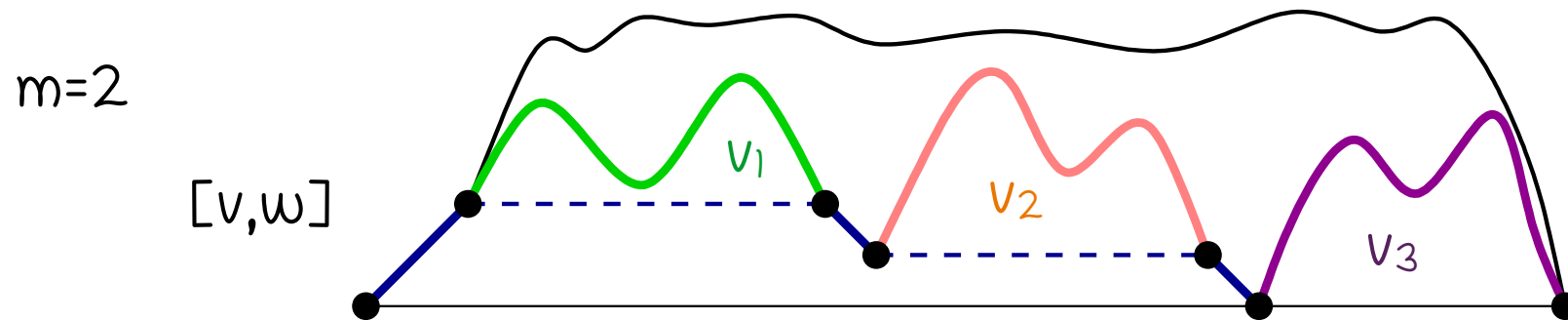
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with $J_0(x) = tx^m$.

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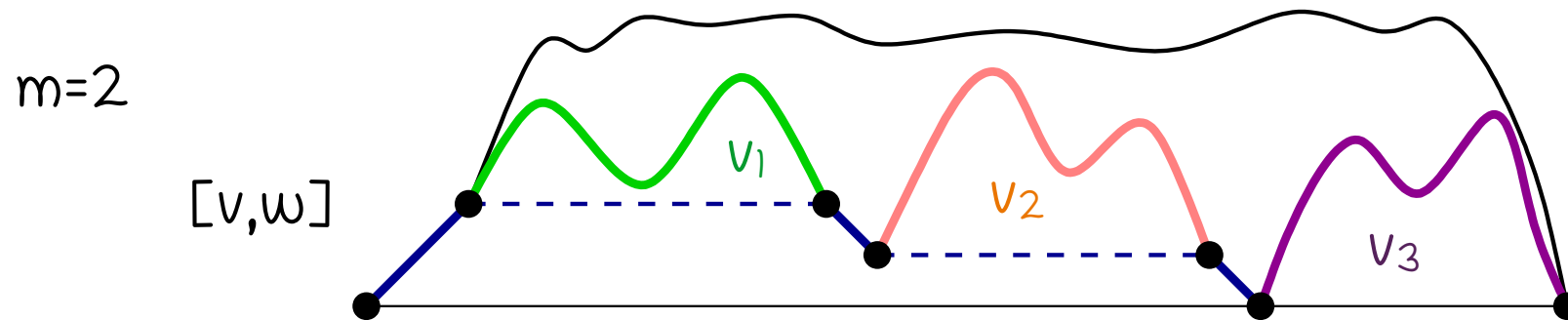
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- An equivalent formulation: if we write $t = z(1 - (m+1)z)^m$, then $J(1)$ becomes a series in z instead of t , which is rational:

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- From small values of m , it seems that the dependency in x can also be parametrized rationally with

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Indeed, with this change of variables, one conjectures:

$$\frac{x^2}{x - 1} J(x) = \frac{zu^{m+2}}{1 - (m + 1)z} \cdot \frac{1}{u - 1}.$$

3/ Checking that the guessed solution is correct

- The equation

$$x^2 J(x) = t (x + x^2 J(x) \Delta)^{(m+2)} \quad (1).$$

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- One has to check a **polynomial identity**:

$$\Lambda^{(m+2)}(1) = u^{m+2} \left(1 - z \sum_{e=0}^m u^e (m+1-e) \right).$$

V. What else?

Final remarks and questions

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- Greedy ν -Tamari order [**Dermenjian 22(a)**]: total number of intervals for ν of size n ?
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- Greedy 1 -Tamari intervals form a subset of ordinary 1 -Tamari intervals: which triangulations do they correspond to?
cf. [**Bernardi & Bonichon 07**]



Thanks for your
attention