

# q-analog of the Markoff injectivity conjecture

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# OUTLINE

1. Markoff injectivity conjecture
2. q-analog of a Markoff Injectivity Conjecture
3. Balanced sequences
4. Main result

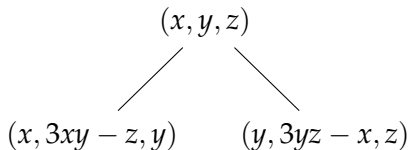
# Markoff Injectivity Conjecture

# MARKOFF NUMBERS

- ▶ **Markoff triple** : A positive solution of the Diophantine equation

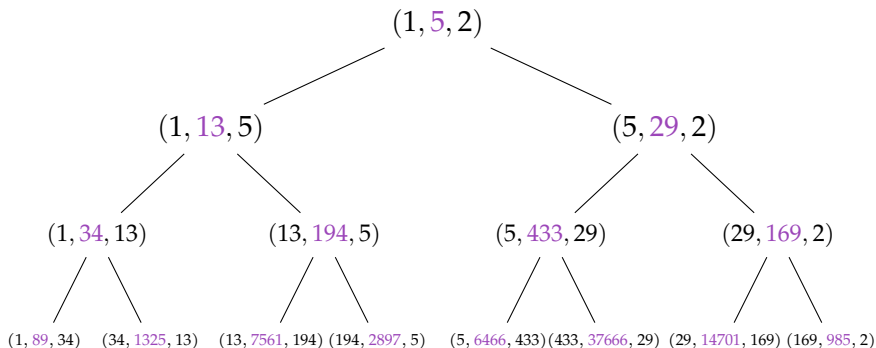
$$x^2 + y^2 + z^2 = 3xyz. \quad (1)$$

- ▶ Positive solutions of Equation 1 can be computed recursively :



- ▶ If  $y > x, y > z$  and  $x \neq z$ , then  $(x, 3xy - z, y) \neq (y, 3yz - x, z)$ ,  $3xy - z > y$  and  $3yz - x > y$ .

# BINARY TREE OF PROPER MARKOFF TRIPLES



# MARKOFF NUMBERS

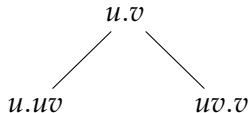
- ▶ **Markoff numbers** : An element of a Markoff triples

1, 2, 5, 13, 29, 34, 89, 169, 194, ...

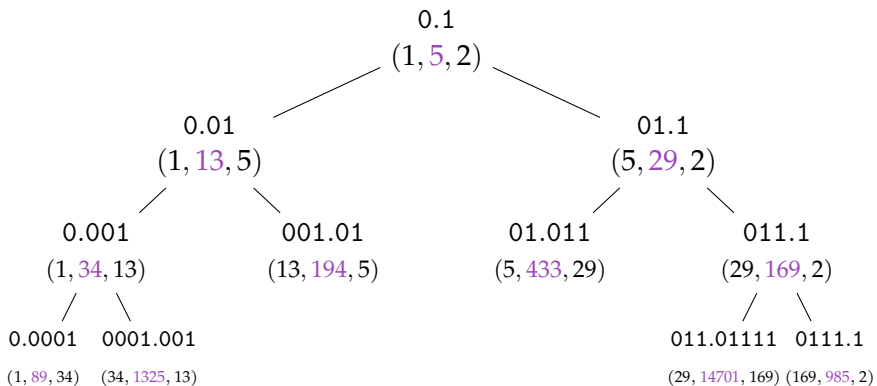
- ▶ OEIS sequence A002559

# CHRISTOFFEL WORDS

- ▶ **Christoffel word** : A word over the alphabet  $\{0, 1\}$  such that  $0, 1, 01$  are Christoffel words and all words satisfying the following binary tree structure :



# BINARY TREE OF PROPER CHRISTOFFEL WORDS AND PROPER MARKOFF NUMBERS





# A MAP BETWEEN WORDS AND MARKOFF NUMBERS

$$\mu : \{0, 1\}^* \rightarrow \mathrm{SL}_2(\mathbb{N})$$

$$\mu(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \mu(1) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

## Proposition (Reutenauer, 2009)

The function  $f$  from the set of *Christoffel words* to the set of *Markoff triples* defines as

$$f(w) = \{\mu(u)_{12}, \mu(w)_{12}, \mu(v)_{12}\},$$

where  $u.v$  is the standard factorization of  $w$ , is a *bijection*.

# MARKOFF INJECTIVITY CONJECTURE

$$\mu : \{0, 1\}^* \rightarrow \mathrm{SL}_2(\mathbb{N})$$

$$\mu(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \mu(1) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

## Conjecture

The map  $w \mapsto \mu(w)_{12}$  is *injective* on the set of Christoffel word.

- ▶ Uniqueness Conjecture, Frobenius conjecture, ...
- ▶ The Markoff Injectivity Conjecture has many other equivalent formulation (Aigner, 2013).

## PARTIAL RESULTS

### Proposition (L. and Reutenauer, 2021)

The map  $w \mapsto \mu(w)_{12}$  is *increasing* over the language of *factors* of a Christoffel word, thus injective.

### Corollary (L. and Reutenauer, 2021)

Moreover, the map is also increasing for all Christoffel words on an *infinite path* in the binary tree of Christoffel words.

Some other partial results were proven recently by Rabideau and Schiffler (2020) and Lagisquet, Pelantová, Tavenas and Vuillon (2021).

# q-analog of a Markoff Injectivity Conjecture

## Q-ANALOG

A **q-analog** is a mathematical expression parametrized by a variable  $q$  that generalizes a known expression.

Examples :

- ▶ q-analog of a **nonnegative integer**  $n$  is

$$[n]_q = 1 + q + \dots + q^{n-1}$$

- ▶ **q-factorial**

$$n!_q = [1]_q [2]_q \dots [n]_q$$

- ▶ **q-binomial coefficients**

$$\binom{n}{k}_q = \frac{n!_q}{(n-k)!_q k!_q}$$

## Q-ANALOG OF RATIONAL NUMBER

- ▶ The  $q$ -analog of a rational number  $\frac{a}{b}$  is a polynomials over  $q$  defined from the continued fraction expansion (Morier-Genoud and Ovsienko, 2020)
- ▶ Their work is based on  $q$ -deformation of the generators of  $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) / \pm Id$

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix} \quad S_q = \begin{pmatrix} 0 & -q^{-1} \\ 1 & 0 \end{pmatrix}$$

## Q-ANALOG OF A MARKOFF NUMBER

The map  $\mu$  can be expressed as of product of the matrix  $R$  and  $S$  :

$$\mu(0) = R^2SR \quad \mu(1) = R^3SR^2SR$$

Hence, the  $q$ -analog of  $\mu(0)$  and  $\mu(1)$  are

$$\mu_q(0) = R_q^2 S_q R_q = \begin{pmatrix} q + q^2 & 1 \\ q & 1 \end{pmatrix},$$

$$\mu_q(1) = R_q^3 S_q R_q^2 S_q R_q = \begin{pmatrix} q + 2q^2 + q^3 + q^4 & 1 + q \\ q + q^2 & 1 \end{pmatrix}$$

and  $\mu_q : \{0, 1\}^* \rightarrow \text{GL}_2(\mathbb{Z}[q^{\pm 1}])$  is a morphism of monoids.

## Q-ANALOG OF A MARKOFF NUMBER

If  $w$  is a **Christoffel word**, then the polynomial  $\mu_q(w)_{12}$  is the **q-analog** of a Markoff number.

**Example :**

$$001 \quad 13 \quad 1 + 2q + 3q^2 + 3q^3 + 3q^4 + q^5$$

**Conjecture**

The map  $w \mapsto \mu_q(w)_{12}$  is *injective* on the set of **Christoffel words**.



## Q-ANALOG OF MARKOFF INJECTIVITY CONJECTURE

## Conjecture

The map  $w \mapsto \mu_q(w)_{12}$  is *injective* on the set of *Christoffel words*.

The  $q$ -analog version is *weaker* than the original Markoff Injectivity Conjecture.

**Example :** 001101 and 010011 have different polynomials

$$\begin{aligned}\mu_q(001101)_{12} = & 1 + 5q + 16q^2 + 37q^3 + 69q^4 + 107q^5 + 143q^6 \\ & + 166q^7 + 169q^8 + 151q^9 + 117q^{10} + 79q^{11} \\ & + 44q^{12} + 19q^{13} + 6q^{14} + q^{15}\end{aligned}$$

$$\begin{aligned}\mu_q(010011)_{12} = & 1 + 5q + 16q^2 + 38q^3 + 70q^4 + 109q^5 + 144q^6 \\ & + 166q^7 + 169q^8 + 150q^9 + 116q^{10} + 77q^{11} \\ & + 43q^{12} + 19q^{13} + 6q^{14} + q^{15}\end{aligned}$$

but when  $q = 1$  :  $\mu(001101)_{12} = \mu(010011)_{12} = 1130$ .

## Q-ANALOG OF MARKOFF INJECTIVITY CONJECTURE

## Conjecture

The map  $w \mapsto \mu_q(w)_{12}$  is *injective* on the set of *Christoffel words*.

The map  $w \mapsto \mu_q(w)_{12}$  from  $\{0, 1\}^*$  to polynomials is **not injective**.

**Example :**

$$\begin{aligned}\mu_q(000111)_{12} &= 1 + 5q + 16q^2 + 38q^3 + 70q^4 + 109q^5 + 145q^6 \\ &\quad + 168q^7 + 171q^8 + 152q^9 + 118q^{10} + 79q^{11} \\ &\quad + 44q^{12} + 19q^{13} + 6q^{14} + q^{15} \\ &= \mu_q(011001)_{12}.\end{aligned}$$

# Balanced sequences

# BALANCED SEQUENCES

- ▶ A biinfinite sequence  $s \in \Sigma^{\mathbb{Z}}$  over a **finite** set  $\Sigma$ .
- ▶ The **language** of  $s$  is the set of factors occurring in  $s$  :

$$\mathcal{L}(s) = \{s_k s_{k+1} \dots s_{k+n-1} \mid k \in \mathbb{Z}, n \geq 0\}$$

# BALANCED SEQUENCES

A sequence  $s \in \Sigma^{\mathbb{Z}}$  is **balanced** if for every  $u, v \in \mathcal{L} \cap \Sigma^n$  and every letter  $a \in \Sigma$ , the number of  $a$ 's occurring in  $u$  and  $v$  differ by at most 1.

In our case,  $\Sigma = \{0, 1\}$ .

Examples :

- ▶ biinfinite periodic  ${}^{\infty}w^{\infty}$  repetition of Christoffel words
- ▶ Sturmian sequences

# FIBONACCI WORD

- ▶ The right-infinite **Fibonacci Word**

$$F = 01001010010010100101 \dots \in \Sigma^{\mathbb{N}}$$

- ▶  $\tilde{F} \cdot 01 \cdot F$

...10100101001001010010 · 01 · 01001010010010100101 ...

- ▶  $\tilde{F} \cdot 10 \cdot F$

...10100101001001010010 · 10 · 01001010010010100101 ...

# FIBONACCI WORDS ARE BALANCED SEQUENCES

►  $\tilde{F} \cdot 01 \cdot F$ :

... 10100101001001010010 · 01 · 01001010010010100101 ...

$n$	$\mathcal{L}(\tilde{F}01F) \cap \Sigma^n$	number of 0's	number of 1's
0	$\{\varepsilon\}$	0	0
1	$\{0, 1\}$	0 or 1	0 or 1
2	$\{00, 01, 10\}$	1 or 2	0 or 1
3	$\{001, 010, 100, 101\}$	1 or 2	1 or 2
4	$\{0010, 0100, 0101, 1001, 1010\}$	2 or 3	1 or 2
5	$\{00100, 00101, 01001, 01010, 10010, 10100\}$	3 or 4	1 or 2

## BINARY BALANCED SEQUENCE

If  $u \in \{0, 1\}^{\mathbb{Z}}$  be a **balanced sequence**, then it falls into exactly one of the following classes :

- (MH<sub>1</sub>)  $u$  is a **purely periodic** word  ${}^{\infty}w^{\infty}$  for some **Christoffel** word  $w$ ,
- (MH<sub>2</sub>)  $u$  is a **generic aperiodic Sturmian** word
- (MH<sub>3</sub>)  $u$  is a **characteristic aperiodic Sturmian** word
- (MH<sub>4</sub>)  $u$  is an **ultimately periodic** word but not purely periodic :

$$\dots xxyxx \dots \text{ or } \dots (ymx)(ymx)(ymy)(xmy)(xmy) \dots ,$$

where  $\{x, y\} = \{0, 1\}$  and  $0m1$  is a **Christoffel word**.



# MARKOFF PROPERTY

We say that a biinfinite word  $s \in \{0, 1\}^{\mathbb{Z}}$  satisfies the **Markoff property** if for any factorization  $s = uxyv$ , where  $\{x, y\} = \{0, 1\}$ , one has

- ▶ either  $\tilde{u} = v$ ,
- ▶ or there is a factorization  $u = u'ym, v = \tilde{m}xv'$ .

## FOUR CLASSES OF BALANCED SEQUENCES

If a biinfinite sequence  $u \in \{0, 1\}^{\mathbb{Z}}$  satisfies the **Markoff property**, then it falls into exactly one of the following classes :

- ( $M_1$ )  $u$  cannot be written as  $u = \tilde{p}xyp$  where  $\{x, y\} = \{0, 1\}$  and the lengths of the Christoffel words occurring in  $u$  are **bounded**;
- ( $M_2$ )  $u$  cannot be written as  $u = \tilde{p}xyp$  where  $\{x, y\} = \{0, 1\}$  and the lengths of the Christoffel words occurring in  $u$  are **unbounded**;
- ( $M_3$ )  $u$  has a **unique factorization**  $u = \tilde{p}xyp$  where  $\{x, y\} = \{0, 1\}$ ;
- ( $M_4$ )  $u$  has **at least two factorizations**  $u = \tilde{p}xyp$  where  $\{x, y\} = \{0, 1\}$ .

# EQUIVALENCE BETWEEN CLASSES

## Proposition (Reutenauer, 2006)

*Let  $u \in \{0, 1\}^{\mathbb{Z}}$  be a balanced sequence. For every  $i \in \{1, 2, 3, 4\}$ ,  $u$  satisfies  $(M_i)$  if and only if  $u$  satisfies  $(MH_i)$ .*

# Main result

# RADIX ORDER

- ▶ Let  $\{0, 1\}$  such that  $0 < 1$ .
- ▶ The **radix order** is defined for every  $u, v \in \{0, 1\}^*$  as

$$u <_{\text{radix}} v \quad \text{if} \quad \begin{cases} |u| < |v| & \text{or} \\ |u| = |v| & \text{and} \quad u <_{\text{lex}} v. \end{cases}$$

# MAIN RESULT

## Proposition (L. and Reutenauer, 2021)

The map  $w \mapsto \mu(w)_{12}$  is *increasing* over the language of *factors* of a Christoffel word, thus injective.

## Theorem

Let  $s \in \{0, 1\}^{\mathbb{Z}}$  be a *balanced sequence* and  $u, v \in \mathcal{L}(s)$  be two *factors* in the language of  $s$ . If  $u <_{\text{radix}} v$ , then  $\mu_q(v)_{12} - \mu_q(u)_{12}$  is a nonzero *polynomial* of indeterminate  $q$  with *nonnegative integer* coefficients.

# MAIN RESULT

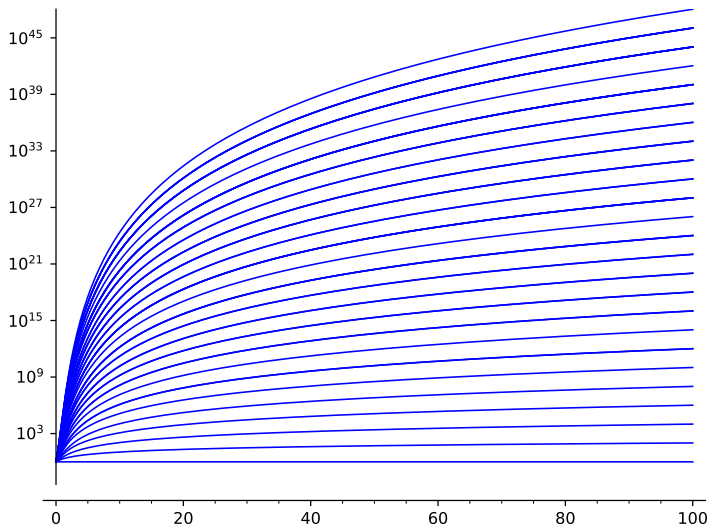
## Theorem

Let  $s \in \{0, 1\}^{\mathbb{Z}}$  be a *balanced sequence* and  $u, v \in \mathcal{L}(s)$  be two *factors* in the language of  $s$ . If  $u <_{\text{radix}} v$ , then  $\mu_q(v)_{12} - \mu_q(u)_{12}$  is a nonzero *polynomial* of indeterminate  $q$  with *nonnegative integer* coefficients.

## Corollary

Let  $s \in \{0, 1\}^{\mathbb{Z}}$  be a *balanced sequence*. For every  $q > 0$ , the map  $\{0, 1\}^* \rightarrow \mathbb{R}$  defined by  $w \mapsto \mu_q(w)_{12}$  is *injective* over the language  $\mathcal{L}(s)$  of *factors* occurring in  $s$ .

THE GRAPH OF  $\mu_q(w)_{12}$  FOR  $0 \leq q \leq 100$  FOR ALL 55  
FACTORS  $w$  OF LENGTH  $|w| < 10$  IN THE FIBONACCI WORD





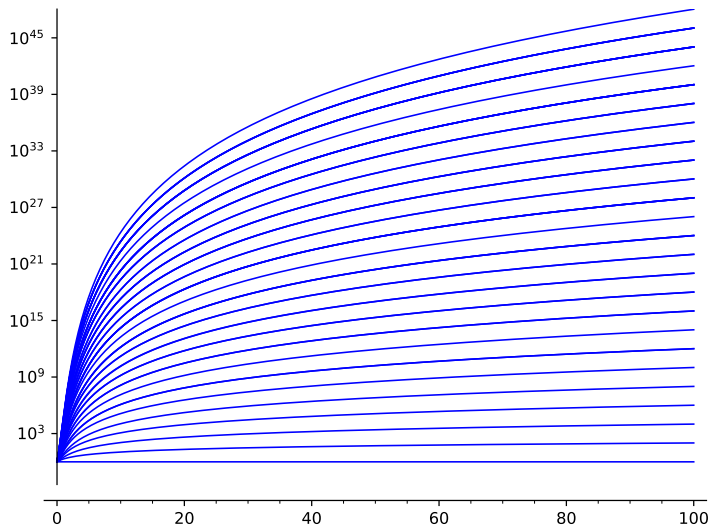
# DIFFERENCE BETWEEN TWO POLYNOMIALS OF SAME DEGREE

The **difference** between two polynomials of the same degree is **very small** since the  $y$  axis uses a **logarithmic scale**.

**Example :**

$$\begin{array}{l} 010 \quad 1 + 2q + 4q^2 + 4q^3 + 3q^4 + 2q^5 + q^6 \\ 100 \quad 1 + 3q + 4q^2 + 4q^3 + 4q^4 + 2q^5 + q^6 \end{array}$$

THE GRAPH OF  $\mu_q(w)_{12}$  FOR  $0 \leq q \leq 100$  FOR ALL 55  
FACTORS  $w$  OF LENGTH  $|w| < 10$  IN THE FIBONACCI WORD



# WHAT DOES IT MEANS TO BE SORTED IN RADIX ORDER?

Proposition (Borel and Reutenauer, 2006)

Let  $w = w_0 <_{lex} w_1 <_{lex} \cdots <_{lex} w_n - 1$  be the conjugates of a *Christoffel word* of length  $n$ . Then, for each  $i = 1, \dots, n - 1$ , one has

$$w_{i-1} = u01v \quad w_i = u10v$$

for some words  $u, v \in \{0, 1\}^*$ .

**Example :** The conjugates of the Christoffel words 00100101 :

00100101  
 00101001  
 01001001  
 01001010  
01010010  
 10010010  
 10010100  
 10100100

# WHAT DOES IT MEANS TO BE SORTED IN RADIX ORDER?

**Example :** The 9 factors of length 8, the largest factor of length 7 and the smallest factors of length 9 of the Fibonacci words :

1010010  
 00100101  
 00101001  
 01001001  
 01001010  
01010010  
 10010010  
 10010100  
 1010010  
10100101  
 001001010

$$\tilde{u}01v \mapsto \tilde{u}10v, \quad w0 \mapsto w1, \quad 1w \mapsto 0w0, \quad 1w \mapsto 0w1$$

where  $u, v, w \in \{0, 1\}^*$  and  $u$  is a prefix of  $v$  or vice versa.

## FACTORS IN RADIX ORDER

## Lemma (Labbé and L., 2021)

Let  $s \in \{0, 1\}^{\mathbb{Z}}$  be a balanced sequence having at least one factorization  $s = \tilde{p}xy p$  where  $\{x, y\} = \{0, 1\}$ . Let  $n \geq 1$  and  $u_0, \dots, u_n$  be the  $n + 1$  factors of length  $n$  of  $s$  such that

$$u_0 <_{\text{lex}} \dots <_{\text{lex}} u_n.$$

If  $w$  is the prefix of length  $n - 1$  of  $p$ , we have

- ▶  $u_0 = 0w$  and  $u_n = 1w$ ,
- ▶ there exists  $i \in \{0, \dots, n - 1\}$  such that  $u_i = \tilde{w}0$  and  $u_{i+1} = \tilde{w}1$ ,
- ▶ for all  $j \in \{0, \dots, n - 1\} \setminus \{i\}$ , there exist prefixes  $x, y$  of  $w$  such that  $u_j = \tilde{x}01y$  and  $u_{j+1} = \tilde{x}10y$ .

## INCREASING OVER SMALL LOCAL CHANGES

## Proposition (Labbé and L., 2021)

For every  $w \in \{0, 1\}^*$ ,

- ▶  $\mu_q(w1)_{12} - \mu_q(w0)_{12}$ ,
- ▶  $\mu_q(0w0)_{12} - \mu_q(1w)_{12}$
- ▶  $\mu_q(0w1)_{12} - \mu_q(0w0)_{12}$

are nonzero *polynomials* with nonnegative coefficients.

## Proposition (Labbé and L., 2021)

Let  $u, v \in \{0, 1\}^*$  such that  $u$  is a prefix of  $v$  or vice versa. Then

$$\mu_q(\tilde{u}10v)_{12} - \mu_q(\tilde{u}01v)_{12}$$

is a nonzero *polynomial* with nonnegative coefficients.

# FACTORS OF STURMIAN WORDS OR ULTIMATELY PERIODIC SEQUENCES

## Proposition (Labbé and L., 2021)

Let  $s \in \{0, 1\}^{\mathbb{Z}}$  be a *balanced sequence* having at least one *factorization*  $s = \tilde{p}xyp$  where  $\{x, y\} = \{0, 1\}$ . Let  $u, v \in \mathcal{L}(s)$  be two factors in the language of  $s$ . If  $u <_{\text{radix}} v$ , then  $\mu_q(v)_{12} - \mu_q(u)_{12}$  is a nonzero *polynomial* with *nonnegative coefficients*.

## Theorem (Labbé and L., 2021)

Let  $s \in \{0, 1\}^{\mathbb{Z}}$  be a *balanced sequence* and  $u, v \in \mathcal{L}(s)$  be two *factors* in the language of  $s$ . If  $u <_{\text{radix}} v$ , then  $\mu_q(v)_{12} - \mu_q(u)_{12}$  is a nonzero *polynomial* of indeterminate  $q$  with *nonnegative integer coefficients*.

## FOUR CLASSES OF BALANCED SEQUENCES

If a biinfinite sequence  $u \in \{0, 1\}^{\mathbb{Z}}$  satisfies the **Markoff property**, then it falls into exactly one of the following classes :

- (M<sub>1</sub>)  $u$  cannot be written as  $u = \tilde{p}xyp$  where  $\{x, y\} = \{0, 1\}$  and the lengths of the Christoffel words occurring in  $u$  are **bounded**;
- (M<sub>2</sub>)  $u$  cannot be written as  $u = \tilde{p}xyxyp$  where  $\{x, y\} = \{0, 1\}$  and the lengths of the Christoffel words occurring in  $u$  are **unbounded**;
- (M<sub>3</sub>)  $u$  has a **unique factorization**  $u = \tilde{p}xyp$  where  $\{x, y\} = \{0, 1\}$ ;
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## BINARY BALANCED SEQUENCE

If  $u \in \{0, 1\}^{\mathbb{Z}}$  be a **balanced sequence**, then it falls into exactly one of the following classes :

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$$\dots xxyxx \dots \text{ or } \dots (ymx)(ymx)(ymy)(xmy)(xmy) \dots ,$$

where  $\{x, y\} = \{0, 1\}$  and  $0m1$  is a **Christoffel word**.

# Back the Markoff Injectivity Conjecture

## LANGUAGE OF ALL BALANCED SEQUENCES

The following statement is **false** : If  $u, v$  in the language of **all balanced sequences** such that  $u <_{radix} v$ , then  $\mu_q(v)_{12} - \mu_q(u)_{12}$  is a nonzero **polynomials** with **nonnegative coefficients**.

**Example** :

$$\begin{aligned}\mu_q(00001)_{12} &= 1 + 4q + 8q^2 + 13q^3 + 16q^4 + 17q^5 + 14q^6 + 10q^7 \\ &\quad + 5q^8 + q^9,\end{aligned}$$

$$\begin{aligned}\mu_q(0111)_{12} &= 1 + 3q + 9q^2 + 16q^3 + 24q^4 + 29q^5 + 29q^6 + 25q^7 \\ &\quad + 18q^8 + 10q^9 + 4q^{10} + q^{11},\end{aligned}$$

and their difference

$$\begin{aligned}\mu_q(00001)_{12} - \mu_q(0111)_{12} &= q - q^2 - 3q^3 - 8q^4 - 12q^5 - 15q^6 - 15q^7 \\ &\quad - 13q^8 - 9q^9 - 4q^{10} - q^{11}\end{aligned}$$

has negative coefficients.

# OPEN PROBLEM

## Conjecture (Markoff Injectivity Conjecture)

The map  $w \mapsto \mu(w)_{12}$  is *injective* on the set of Christoffel word.

## Conjecture (q-analog of Markoff Injectivity Conjecture)

The map  $w \mapsto \mu_q(w)_{12}$  is *injective* on the set of Christoffel word.