

Young tableaux with periodic walls: counting with the density method

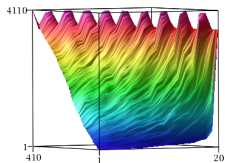
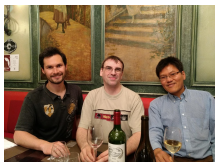
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GT Combinatorics and Interactions, LaBRI
November 8th, 2021

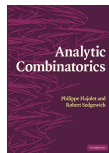
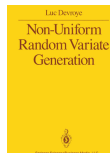
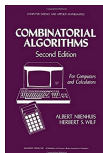
3	10	5	6	12	16	13	14
1	2	4	7	8	9	11	15



Uniform random generation of combinatorial structures

Many approaches:

- ad hoc methods (& general principles: bijections, rejection, (un)ranking, etc.)
- Markov chain Monte Carlo algorithms, e.g. coupling from the past [Propp Wilson 1998]
- recursive method [Nijenhuis Wilf 1975, Flajolet Zimmermann Van Cutsem 1994, Denise Dutour Zimmermann 1998]
 \rightsquigarrow packages in MuPAD/SageMath/Maple
- generating trees [West 1990, Dulucq Gire Guibert 1996, Barucci Del Lungo Pergola Pinzani 1998, Banderier Bousquet-Mélou Denise Flajolet Gardy Gouyou-Beauchamps 1998...]
- Boltzmann method [Duchon Flajolet Louchard Schaeffer 2002, Fusy Pivoteau Salvy Soria Bodini Ponty Dovgal Bendkowsky Dien Papin Bacher Sportiello Stuffer...]: the cherry on the cake of the symbolic method!
- density method \rightsquigarrow **this talk!**



Part I:

Enumerative and bijective results

Young tableaux with local decreases

7	18	19	12	21	20	17
2	6	8	9	10	14	16
1	3	4	5	11	13	15

We consider **Young tableaux** in which some pairs of (horizontally or vertically) consecutive cells are allowed to have **decreasing labels**. Places where a decrease is allowed (but not compulsory) are drawn by a red edge, which we call a **“wall”**.

Nice formulas for some specific tableaux of shape $n \times 2$:

Young tableaux with local decreases

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Nice formulas for some specific tableaux of shape $n \times 2$:

- no walls:

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Nice formulas for some specific tableaux of shape $n \times 2$:

- no walls: $\frac{1}{n+1} \binom{2n}{n}$

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14	12
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3	11
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4	6
2	5
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Nice formulas for some specific tableaux of shape $n \times 2$:

- no walls: $\frac{1}{n+1} \binom{2n}{n}$
- walls everywhere:

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- horizontal walls everywhere:

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- horizontal walls everywhere in 2^{nd} col.:

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- vertical walls everywhere:

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- all k vertical walls:

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- all k vertical walls: $\frac{1}{n+1} \binom{n+1}{k} \binom{2n}{n}$ (We give 2 proofs 😊)

Proof #1: bijection with paths

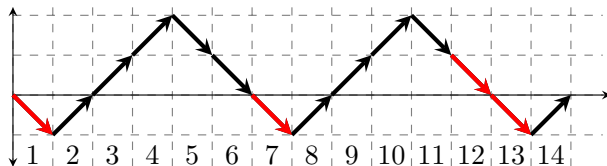
Theorem

The number of $n \times 2$ Young tableaux \mathcal{Y} with k vertical walls is equal to

$$v_{n,k} = \frac{1}{n+1} \binom{n+1}{k} \binom{2n}{n}.$$

Proof: (part 1) Bijection with Dyck bridges on \mathbb{Z} : steps ± 1 , length $2n$, k marked down steps:

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10	12
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Proof #1: bijection with paths

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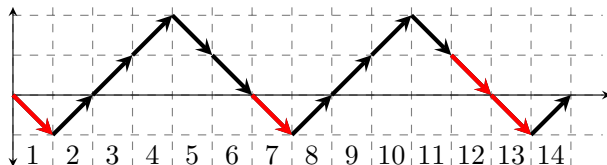
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- The k th step is an up step iff the entry k appears in the first column of \mathcal{Y} ; otherwise it is a down step.
- If the k th down step is in a row with a wall we color it.

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Theorem

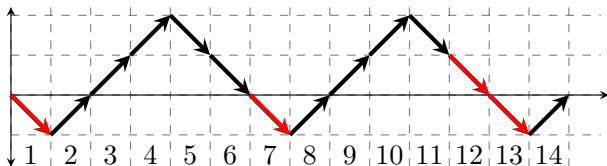
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 - If the k th down step is in a row with a wall we color it.
- $\Rightarrow v_{n,k}$ counts the **number of paths** with exactly k colored steps.



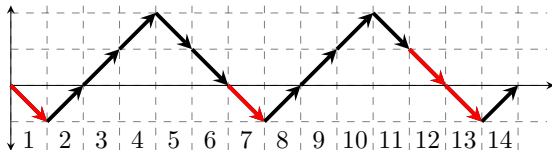
Proof #1: bijection with paths and Chung–Feller

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Proof (part 2):

- A down step below 0 is always colored because a wall has to be involved, a down step above 0 has a choice.
- By [Chung–Feller 49] the number of Dyck bridges of length $2n$ with i down steps below 0 is independent of i and equal to $\text{Cat}(n) = \frac{1}{n+1} \binom{2n}{n}$.

$$\Rightarrow v_{n,k} = \sum_{i=0}^k \binom{n-i}{k-i} \text{Cat}(n) = \binom{n+1}{k} \text{Cat}(n). \quad \square$$

Proof #2: bijection to leaf-marked binary trees

Theorem

The GF for a fixed size n and an arbitrary number of walls is

$$v_n(u) := \sum_{k=0}^n v_{n,k} u^k = \text{Cat}(n) ((1+u)^{n+1} - u^{n+1}).$$

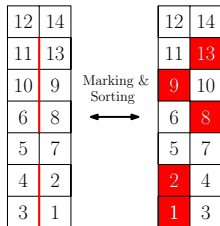
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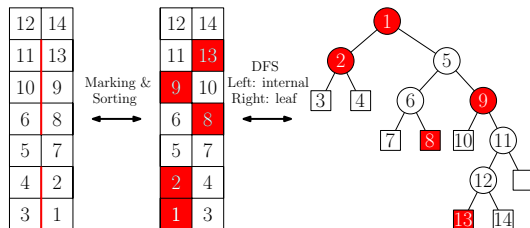


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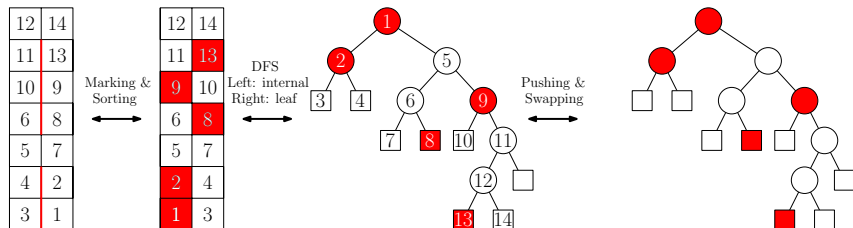


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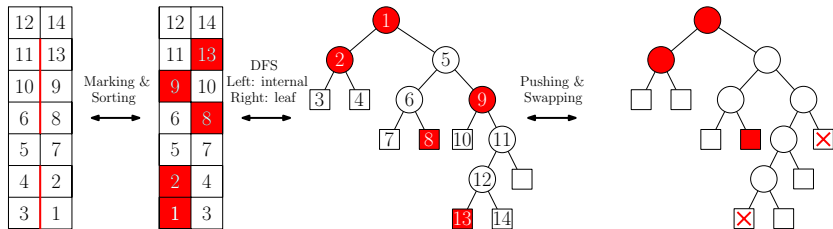


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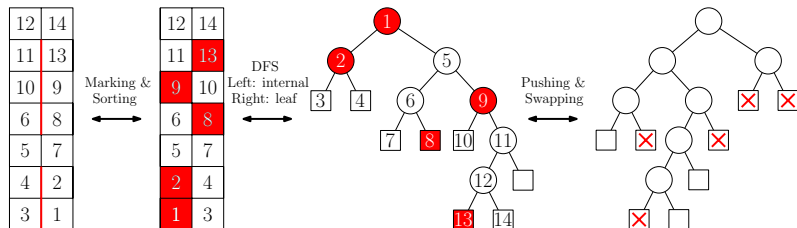


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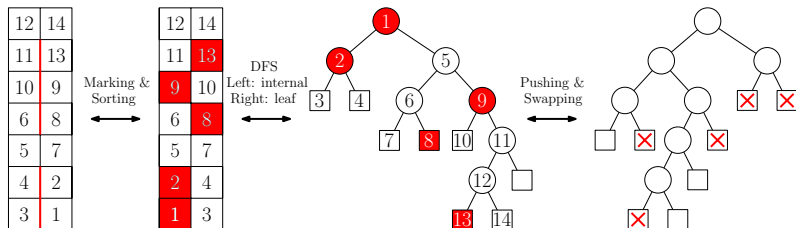


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Open problem: combinatorial explanation of $\text{Cat}(k-1) \mid v_{n,k}$?

Closed form of $v_{n,k}$ also proves $\sum_{n \geq 0} v_{n,k} z^n = \text{Cat}(k-1) \frac{z^{k-1}}{(1-4z)^{(2k-1)/2}}$.

Long walls with small holes: hook-length type formulas

Holes of size 1 on the border

13	14	16	17	19	20	21	25	27
11	2	10	12	15	18	6	23	26
4	1	8	5	7	9	3	22	24

\longleftrightarrow \longleftrightarrow \longleftrightarrow \longleftrightarrow

λ_1 λ_2 λ_3 λ_4

Theorem

The number of $n \times m$ Young tableaux of size mn with k walls from column 1 to $m - 1$ at distance $0 < d_i := \sum_{j=1}^i \lambda_j < n$, $i = 1, \dots, k$ with $h_i < h_{i+1}$ is equal to

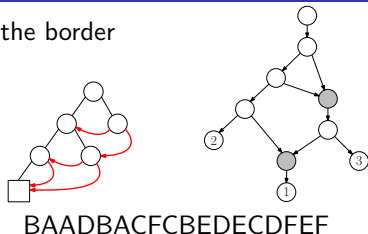
$$\frac{(m-1)!}{(mn+m-1)_{m-1}} \left(\prod_{i=1}^{k+1} \prod_{j=1}^{m-2} \binom{\lambda_i + j}{j}^{-1} \right) \left(\prod_{i=1}^{k+1} \binom{md_i + m - 1}{\lambda_i, \dots, \lambda_i} \right),$$

where the multinomial coefficients contain $m - 1$ λ_i 's.

Larger holes lead to unusual asymptotics

The “simplest case” of holes of size 2 on the border

6	10	14	15	17	18
3	5	9	12	13	16
2	1	7	4	11	8



Theorem

The number f_n of such Young tableaux of size $n \times 3$ satisfies

$$f_n = \Theta \left(n! 12^n e^{a_1(3n)^{1/3}} n^{-2/3} \right),$$

where $a_1 \approx -2.338$ is the largest root of the Airy function of the first kind.

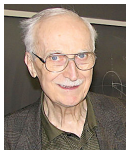
- Bijections to phylogenetic networks, special words with n distinct letters, and related to compacted trees (special DAGs) [Fuchs–Yu–Zhang 21]
- General method to prove **stretched exponentials** in bivariate recurrences [Elvey Price–Fang–Wallner 21]. Here:

$$y_{n,k} = y_{n,k-1} + (2n + k - 1)y_{n-1,k} \quad \text{and} \quad f_n = y_{n,n}.$$

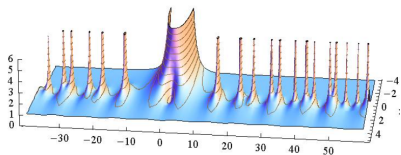
Part II:

The density method

- * far origins in poset theory (volume of polytopes, log-concavity) [Stanley 1981]
- * avatars in number theory [Zagier, Beukers Kolk Calabi 1993, Elkies 2003]
- * applied to square Young tableaux [Barishnikov 2001]
and variants of alternating permutations [Baryshnikov Romik 2010, Stanley 2010]
- * generalized to further posets & random generation [Banderier Marchal Wallner 2016–2021]



Values of the zeta function



$$\zeta(s) = \sum_{k \geq 1} \frac{1}{k^s} \quad \rightarrow \quad \zeta(s) \left(1 - \frac{1}{2^s}\right) = \sum_{k \geq 0} \frac{1}{(2k+1)^s}$$

$$S(2) = \sum_{k \geq 0} \frac{1}{(2k+1)^2} = \sum_{k \geq 0} \int_0^1 \int_0^1 (xy)^{2k} = \int_0^1 \int_0^1 \frac{dx dy}{1 - (xy)^2}$$

Change of variable $x = \frac{\sin u}{\cos v}$ and $y = \frac{\sin v}{\cos u}$.

The integration domain becomes the triangle $T = \{u > 0, v > 0, u + v < \pi/2\}$.

$$S(2) = \int_T du dv = \pi^2/8$$

n even: $S(n) = \text{vol}(\text{polytope of dimension } n) = (\pi/2)^n / n! A(n)$

$A(n) = \#$ alternating permutations of length n .

↪ Kontsevich–Zagier periods / Hilbert 3rd problem

Uniform random generation and enumeration

6	15	16
1	13	14
8	10	18
3	9	12
4	7	17
2	5	11

This example is “without loss of generality”
(i.e., our method works also
for non-periodic shapes). 😊

How to generate/enumerate such tableaux? Brute-force is hopeless!

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Solution = use our density method!

The density method will give thousands of coefficients in a few seconds.

The number of tableaux of size $2n \times 3$ is $f_n = (6n + 1)! \int_0^1 p_n(z) dz$, with

$$p_{n+1}(z) = \int_0^z \frac{1}{24} (z-1)(x-z)(3x^3 - 7x^2z - xz^2 - z^3 - 2x^2 + 4xz + 4z^2) p_n(x) dx.$$

$\{f_n\}_{n \geq 0} = \{1, 12, 8550, 39235950, 629738299350, 26095645151941500, 2323497950101372223250, 392833430654718548673344250, 115375222087417545717234273063750, 55038140590519890608190921051205837500, \dots\}$.

From tableaux to tuples of real numbers, and polytopes

6	15	16
1	13	14
8	10	18
3	9	12
4	7	17
2	5	11

7	16	17
2	14	15
9	11	19
4	10	13
5	8	18
3	6	12
	1	

.74	.96	.97
.25	.94	.95
.85	.91	.99
.42	.90	.93
.54	.82	.98
.35	.57	.92
	.06	

S	Z	W
R	Y	V
	X	

S	<	Z	<	W
∨		∨		∨
R	<	Y	<	V
		∨		
		X		

Left: $2n \times 3$ Young tableau with walls.

Centre: A related tableau with one more cell (removing this cell + relabel: bijection with left tableau).

Our algorithm generates **real numbers** between 0 and 1, with same relative order. All possible values = a polytope $\mathcal{P} \in [0, 1]^{6n+1}$.

Right: The “**building block**” of 7 cells. Each polyomino is made of the overlapping of n such building blocks.

Density method: key ideas

- geometric point of view:

Associate with a poset of size N its *order polytope* \mathcal{P} (it is a subset of $[0, 1]^N$).
Generate a random element of \mathcal{P} slice by slice using conditional densities.
In the present example, $N = 6n + 1$ and the slices are the building blocks of size 6 (except for the first one).

- sequence of densities:

sequence of polynomials $p_n(x)$, defined by the following recurrence (which in fact encodes the full structure of the problem, building block after building block):

$p_0 = 1$ and by induction,

$$p_{n+1}(z) = \int_{0 < x < z} \int_{x < y < z} \int_{0 < r < y} \int_{r < s < z} \int_{z < w < 1} \int_{y < v < w} p_n(v) dv dw ds dr dy dx.$$

S	Z	W
R	Y	V
	X	

S	$<$	Z	$<$	W
\vee		\vee		\vee
R	$<$	Y	$<$	V
		\vee		
		X		

The density method algorithm

.74	.96	.97
.25	.94	.95
.85	.91	.99
.42	.90	.93
.54	.82	.98
.35	.57	.92
	.06	

S	<	Z	<	W
V		V		V
R	<	Y	<	V
		V		
		X		

- Initialization:** Precompute the polynomials $p_0(z), \dots, p_n(z)$.
Label the building blocks from $k = n - 1$ to $k = 0$ (top to bottom).
Start at the top, i.e. $k := n - 1$.
Put into the top cell Z a random number z with density $p_n(z) / \int_0^1 p_n(t) dt$.
- Filling:** Now that Z is known, put into the cells X, Y, R, S, V, W random numbers x, y, r, s, v, w with conditional density

$$g_{k,z}(x, y, r, s, v, w) := \frac{1}{p_{k+1}(z)} p_k(x) \mathbf{1}_{\mathcal{P}_k},$$

where $\mathbf{1}_{\mathcal{P}_k}$ is the indicator function of the k -th building block (with value z in cell Z):

$$\mathbf{1}_{\mathcal{P}_k} := \mathbf{1}_{\{0 \leq x \leq y \leq z, 0 \leq r \leq y, r \leq s \leq z, z \leq w \leq 1, y \leq v \leq w\}}.$$

- Iteration:** Consider X as the Z of the next building block. Set $k := k - 1$ and go to step 2 (until $k = 0$).

A very efficient algorithm!

Theorem

The density method algorithm is a *uniform random generation algorithm* with *quadratic time complexity* (including precomputations).

Proof

Our algorithm yields a tuple $\mathbf{x} := (x_i, y_i, r_i, s_i, v_i, w_i)_{0 \leq i \leq n}$ with density

$$\begin{aligned} & \frac{p_n(x_n)}{\int_0^1 p_n(t) dt} \prod_{i=1}^n g_{n-i, x_{n-i+1}}(x_{n-i}, y_{n-i}, r_{n-i}, s_{n-i}, v_{n-i}, w_{n-i}) \\ &= \frac{p_n(x_n)}{\int_0^1 p_n(t) dt} \prod_{k=0}^{n-1} \frac{p_k(x_k) \mathbf{1}_{\mathcal{P}_{x_k}}}{p_{k+1}(x_{k+1})} \underset{\text{telescopic product!}}{=} \frac{p_0(x_0) \mathbf{1}_{\mathcal{P}}}{\int_0^1 p_n(t) dt} = \frac{\mathbf{1}_{\mathcal{P}}}{\int_0^1 p_n(t) dt} \quad (\text{as } p_0 = 1), \end{aligned}$$

where $\mathbf{1}_{\mathcal{P}_{x_k}}$ is as in the algorithm above the indicator function of the k -th block.
 \int (any density) = 1 implies:

$$\text{vol}(\mathcal{P}) = \int_{[0,1]^{6n+1}} \mathbf{1}_{\mathcal{P}} d\mathbf{x} = \int_0^1 p_n(t) dt.$$

A very efficient algorithm!

end of proof (uniformity & complexity):

Now if we choose a random uniform element in $[0, 1]^{6n+1}$, the probability that it belongs to \mathcal{P} is

$$\int_{[0,1]^{6n+1}} \mathbf{1}_{\mathcal{P}} \, d\mathbf{x}.$$

This is also the probability that a **random uniform** filling of our building block is correct (i.e., respects the order constraints).

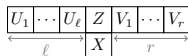
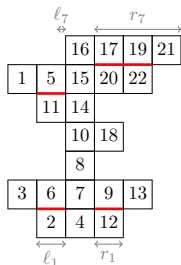
This implies that $f_n = (6n + 1)! \int_0^1 p_n(t) dt = |\mathcal{P}|! \operatorname{vol}(\mathcal{P})$.

Each step = computation and evaluation of the associated polynomial $p_n(z)$ (of degree proportional to n) \Rightarrow **quadratic time complexity** and **quadratic space**. \square

Jenga tableaux and the density method



Jenga! = Construct! in Swahili.



Given a shape encoded by $(\ell_i, r_i)_{i \in \mathbb{N}}$, what is the number of tableaux with n lines?

$$f_n = \left(\sum_{i=1}^n (\ell_i + r_i + 1) \right)! \int_0^1 p_n(x) dx.$$

$$p_n(z) = \int_{z < v_1 < 1} \dots \int_{v_{r-1} < v_r < 1} \int_{0 < u_\ell < z} \dots \int_{0 < u_1 < u_2} \int_{0 < x < z} p_{n-1}(x) dx du_1 \dots du_\ell dv_r \dots dv_r$$

$$= \frac{z^{\ell_n} (1-z)^{r_n}}{\ell_n! r_n!} \int_0^z p_{n-1}(x) dx \quad \text{with} \quad p_1(z) = \frac{z^{\ell_1} (1-z)^{r_1}}{\ell_1! r_1!}.$$

Special Jenga shapes

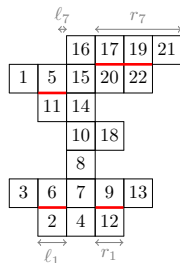
Theorem (1-periodic Jenga tableaux are D-finite)

The bivariate GF $P(t, z) = \sum_{n \geq 1} p_n(z) t^n$ is D-finite in t and z .

Proposition

For $r_i = 0, \forall i \geq 1$, the number f_n of Jenga tableaux is

$$f_n = \frac{(\sum_{i=1}^n (\ell_i + 1))!}{\prod_{i=1}^n \ell_i! (\sum_{j=1}^i (\ell_j + 1))}.$$



Proposition

For Jenga tableaux with period p , $L := \sum_{i=1}^p \ell_i$, and $(r_i)_{i=0}^p = (0, \dots, 0)$, one has:

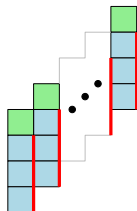
$$f_{kp+m} = f_m \left(\frac{(L+p)^L}{\prod_{i=1}^p \ell_i!} \right)^k \prod_{\substack{j=1 \\ j \neq \ell_1 + \dots + \ell_i + i}}^{L+p} \frac{\Gamma\left(k + \frac{j+m}{L+p}\right)}{\Gamma\left(\frac{j+m}{L+p}\right)}.$$

Accordingly, the GF of such tableaux is the **sum of p hypergeometric functions**.

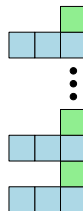
Skew tableaux with walls



Building block



Skew shape



Jenga shape

Proposition

The number of above skew tableaux with n columns of height h is

$$f_n = \left(\frac{h^{h-2}}{(h-2)!} \right)^n \prod_{j=1}^{h-2} \frac{\Gamma\left(n + \frac{j}{h}\right)}{\Gamma\left(\frac{j}{h}\right)}.$$

Proof: Use a bijection between this class and periodic Jenga tableaux of period $p = 2$, $\ell_1 = h - 2$, $\ell_2 = 0$, and $r_i = 0$. □

A classification of 2×2 periodic shapes

A **periodic shape** is the concatenation of n copies of a building block \mathcal{B} of j cells:

$$\mathcal{Y} = \mathcal{B}^n.$$

A **tableau \mathcal{Y} with periodic walls** is a periodic shape filled with all integers from $\{1, \dots, |\mathcal{B}|n\}$ respecting the induced order constraints.

$$\mathcal{B} = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$
$$\mathcal{B}^4 = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 3 & 10 & 5 & 6 & 12 & 16 & 13 & 14 \\ \hline 1 & 2 & 4 & 7 & 8 & 9 & 11 & 15 \\ \hline \end{array}$$

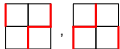



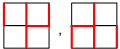


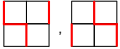











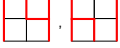

There are a priori $2^6 = 64$ shapes, but some are in bijection (e.g., turn by 180 degrees and reverse labels). It turns out that it leads to 32 different sequences.

We now characterize all 2×2 shapes according to the nature of the counting sequence/generating function, which is either

- “simple” hypergeometric
- hypergeometric,
- algebraic,
- D-algebraic and beyond.

“Simple” hypergeometric cases


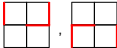

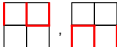



= cases where walls split tableaux into **independent regions** \Rightarrow product formulas

Class	Shape	Formula	Class	Shape	Formula	Class	Shape	Formula
P1		$4 \frac{(4n)!}{24^n}$	P6		$\frac{(4n)!}{6^n}$	P13		$\frac{3}{2} \frac{(4n)!}{3^n}$
P2		$\frac{(4n)!}{12^n}$	P7		$3 \frac{(4n)!}{6^n}$	P14		$\frac{(4n)!}{2^n}$
P3		$3 \frac{(4n)!}{12^n}$	P8		$\frac{8}{5} (4n)! \frac{5^n}{24^n}$	P15		$2 \frac{(4n)!}{2^n}$
P4		$\frac{(4n)!}{8^n}$	P9		$\frac{(4n)!}{4^n}$	P16		$\frac{(4n)!}{(2n)! 2^n}$
P5		$4 \frac{(4n)!}{8^n}$	P10		$2 \frac{(4n)!}{4^n}$	P17		$2 \frac{(4n)!}{(2n)! 2^n}$
			P11		$4 \frac{(4n)!}{4^n}$	P18		$\frac{(4n)!}{(2n)!}$
			P12		$\frac{(4n)!}{3^n}$	P19		$(4n)!$

Proofs: Choose and distribute elements according to constraints.

Hypergeometric cases

= cases with **uniquely determined minimum or maximum**



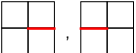
Class	Shape	Sequence	OEIS
H1		$\prod_{i=1}^n (4i-1)(4i-3)$	A101485
H2		$\prod_{i=1}^n (2i-1)(4i-1)$	A159605
H3		$2^{n+1} n! \prod_{i=1}^n (4i-3)$	$2^{n+1} \cdot$ A084943
H4		$\binom{4n}{n} \prod_{i=1}^n (3i-1)$	$\binom{4n}{n} \cdot$ A008544
H5		$\binom{4n}{n} \prod_{i=1}^n (3i-2)$	$\binom{4n}{n} \cdot$ A007559
H6		$2^n n! \prod_{i=1}^n (4i-3)$	$n! \cdot$ A084948
H7		$\prod_{i=1}^n (2i-1)(4i-1)$	A159605

Proofs:

- Models H1–H5: variants of Jenga tableaux with $r_i = 0$ for all i
- Models H6–H7: recursively decompose with respect to the location of the unique minimum or maximum.

Algebraic cases

= cases with **no vertical walls**

Class	Shape	Sequence	OEIS
A1		$\text{Cat}(2n) = \frac{1}{2n+1} \binom{4n}{2n}$	A048990
A2		$\binom{4n}{2n}$	A001448
A3		$2^{2n+1} \text{Cat}(n) - \text{Cat}(2n+1)$	A079489

Proofs:

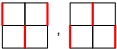


- Models A1 and A2: Use bijections to Dyck bridges.
- Model A3: Decomposing at the first wall that cannot be removed gives

$$f_n = \text{Cat}(2n) + \sum_{i=1}^n \text{Cat}(2i-1) f_{n-i},$$

which we then solve with generating functions.

D-algebraic cases?

≈ cases with a zig-zag-like pattern

Class	Shape	GF	OEIS	Example																
Z1		D-algebraic, and not D-finite: $\frac{\cos(t/\sqrt{2})^2 + \cosh(t/\sqrt{2})^2}{2 \cos(t/\sqrt{2}) \cosh(t/\sqrt{2})}$	related to A211212	<table border="1" data-bbox="964 244 1219 308"> <tr><td>12</td><td>16</td><td>6</td><td>15</td><td>13</td><td>14</td><td>7</td><td>10</td></tr> <tr><td>8</td><td>3</td><td>5</td><td>9</td><td>11</td><td>2</td><td>4</td><td>1</td></tr> </table>	12	16	6	15	13	14	7	10	8	3	5	9	11	2	4	1
12	16	6	15	13	14	7	10													
8	3	5	9	11	2	4	1													
Z2		open problem!	—	<table border="1" data-bbox="964 365 1219 429"> <tr><td>3</td><td>5</td><td>8</td><td>9</td><td>11</td><td>13</td><td>14</td><td>15</td></tr> <tr><td>2</td><td>10</td><td>4</td><td>7</td><td>1</td><td>16</td><td>6</td><td>12</td></tr> </table>	3	5	8	9	11	13	14	15	2	10	4	7	1	16	6	12
3	5	8	9	11	13	14	15													
2	10	4	7	1	16	6	12													
Z3		open problem!	—	<table border="1" data-bbox="964 482 1219 547"> <tr><td>2</td><td>4</td><td>5</td><td>8</td><td>11</td><td>12</td><td>14</td><td>15</td></tr> <tr><td>13</td><td>3</td><td>16</td><td>7</td><td>9</td><td>6</td><td>10</td><td>1</td></tr> </table>	2	4	5	8	11	12	14	15	13	3	16	7	9	6	10	1
2	4	5	8	11	12	14	15													
13	3	16	7	9	6	10	1													

Proof for Z1: A permutation (a_1, \dots, a_n) is an **alternating permutation of type (k_1, \dots, k_m)** if

$$a_1 < \dots < a_{k_1} > a_{k_1+1} < \dots < a_{k_1+k_2} > a_{k_1+k_2+1} < \dots < a_n.$$

Then, $k_i = 1$ gives classical alternating permutations;

while $k_1 = 3$, $k_2 = \dots = k_n = 4$, and $k_{n+1} = 1$ gives Z1.

A generalization of [Carlitz 73] then leads to

$$F(t) = \frac{E_{4,3}(t)E_{4,1}(t)}{E_{4,0}(t)} + E_{4,0}(t) \quad \text{where} \quad E_{k,r}(t) = \sum_{n \geq 0} (-1)^n \frac{t^{nk+r}}{(nk+r)!}. \quad \square$$



Leonard Carlitz
(1907-1999)
771 articles!

Conclusion

- 3 ways to enumerate and generate Young tableaux with walls: hook-length type formulas, bijections, density method.
- Approach different from [Greene Nijenhuis Wilf 84]. They used the existence of a simple product formula (hook-length formula).
- Brute-force generation \rightarrow exponential cost. Generation via our density method $\rightarrow O(n^2)$ cost.
- A field to explore: examine more families of posets (e.g., permutations, Young tableaux, increasing trees, urn models in [Banderier Marchal Wallner 20]).
- Asymptotics? D-finite? D-algebraic? Links with other objects?

3	5	8	9	11	13	14	15
2	10	4	7	1	16	6	12

$$\Theta(n! C^n e^{a_1 n^\sigma} n^\alpha) ?$$

