Local parking algorithms on $\mathbb{Z}$

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GT Combinatoire et Interactions, LaBRi, 2 mai 2022

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A word of warning

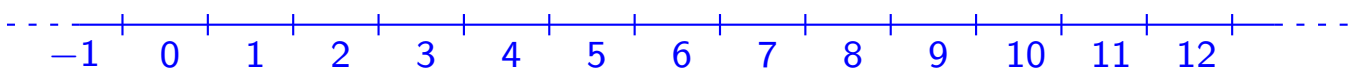
This is a work in progress. Bibliographical research was done, but some constructions and results may already be in the literature somewhere...
If you notice such an occurrence, please forgive my ignorance and let me know!

Classical Parking Functions

Classical parking procedure ([Konheim-Weiss '66]).

- r cars want to park on an empty street where the spots are labeled by $\mathbb{Z}$.
- The cars arrive successively, and the ith car has a preferred spot a_i.
- If the spot is available, it parks there.
- If not, it parks in the *nearest available spot on the right*.

Example. $a_1a_2a_3a_4a_5a_6a_7 = 3525895$



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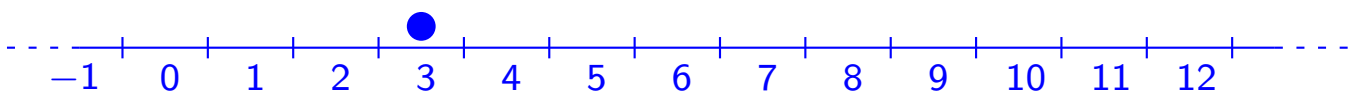

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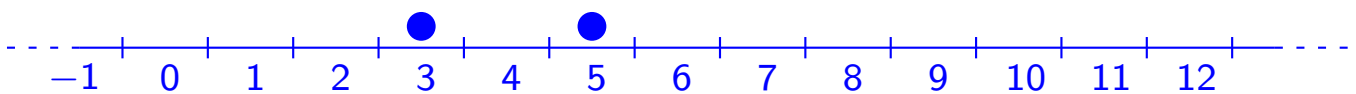

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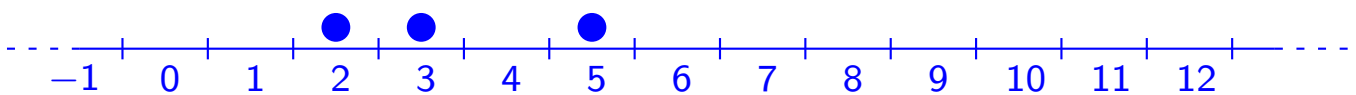

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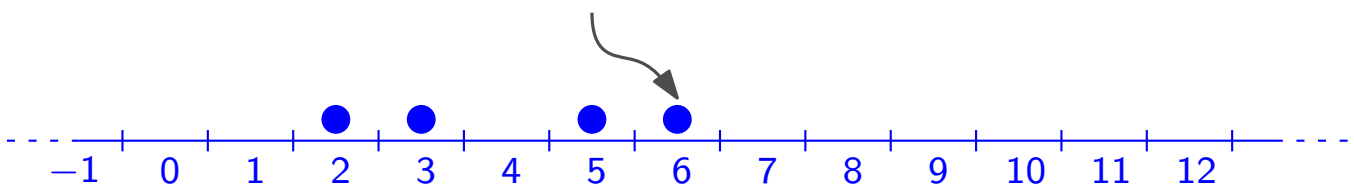

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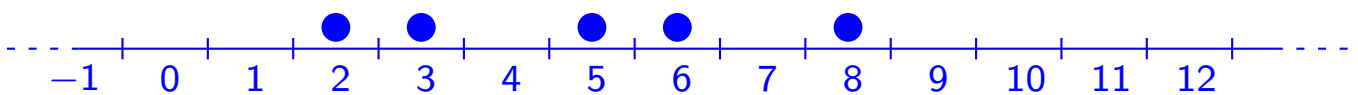

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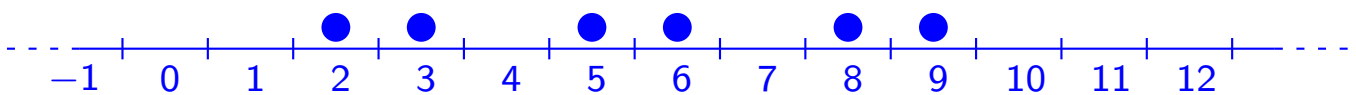

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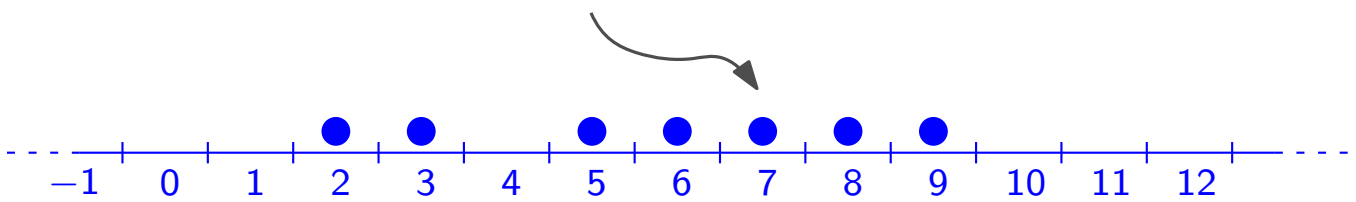

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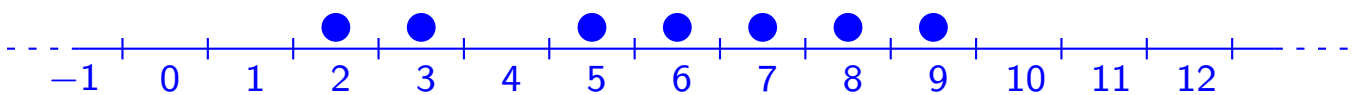

Definition. A word $a_1a_2 \dots a_r$ is called a parking word (or function) if the parking procedure ends up with all spots $1, \dots, r$ occupied.

Example.

$(r = 1)$ 1
$(r = 2)$ $11, 12, 21$
$(r = 3)$ $111, 112, 113, 121, 122, 123, 131, 132, 211, 212, 213, 221, 231, 311, 312, 321$

Classical Parking Functions

Proposition. The number of parking functions of length r is $(r+1)^{r-1}$

Proof. Elegant proof by Pollak (c. '74), which we generalize later. □

Proposition. $a_1 a_2 \cdots a_r$ is a parking function
$\Leftrightarrow$ for $k = 1, \ldots, r$, there are at least k indices i such that $1 \leq a_i \leq k$.

Corollary (Abelian property). As a consequence, if $a_1 a_2 \cdots a_r$ is a parking function then so is $a_{\sigma(1)} \cdots a_{\sigma(r)}$ for any permutation σ of $\{1, \ldots, r\}$.

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Occurrences of parking functions (see various talks by R. Stanley, survey by C. Yan).

- Minimal factorizations of the long cycle $(1, 2, \ldots, r+1)$ in transpositions.
- Maximal chains in the lattice of noncrossing partitions
- Diagonal harmonics for the symmetric group.
- Regions of the Shi hyperplane arrangement $\{x_i - x_j = 0, 1\}$
- Volume of the Pitman-Stanley polytope.

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Generalizations: **u**-parking functions, G-parking functions...

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In more mathematical terms, a parking procedure will be modeled by a function

$$\mathcal{P} : \mathbb{Z}^* = \{\text{Words over } \mathbb{Z}\} \to \text{Finset}(\mathbb{Z}) = \{\text{Finite subsets of } \mathbb{Z}\}.$$

"List of wanted spots" "Set of occupied spots"

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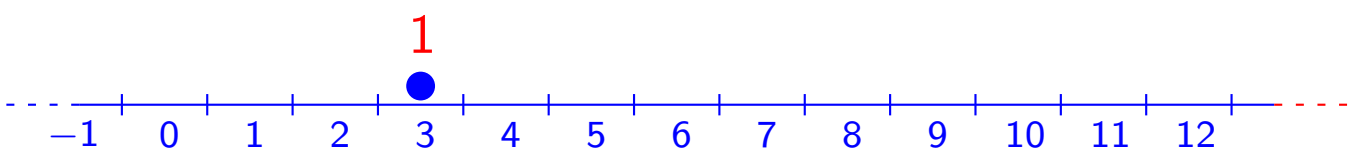

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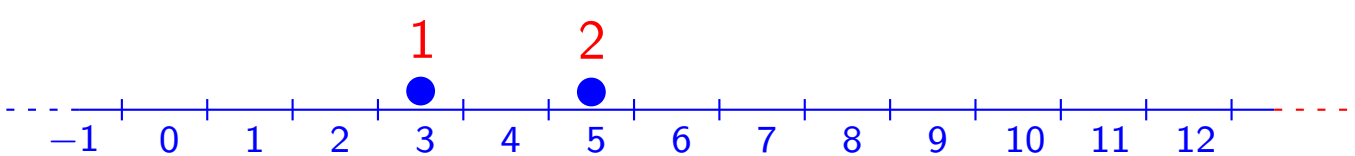

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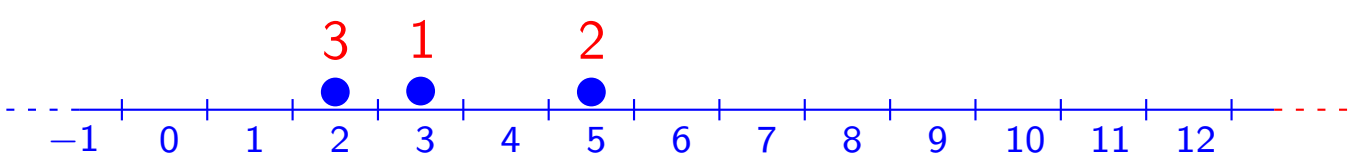

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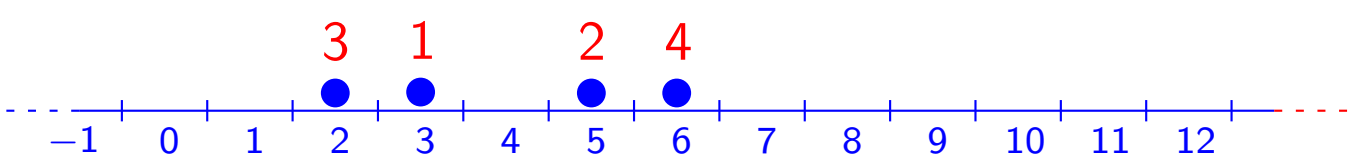

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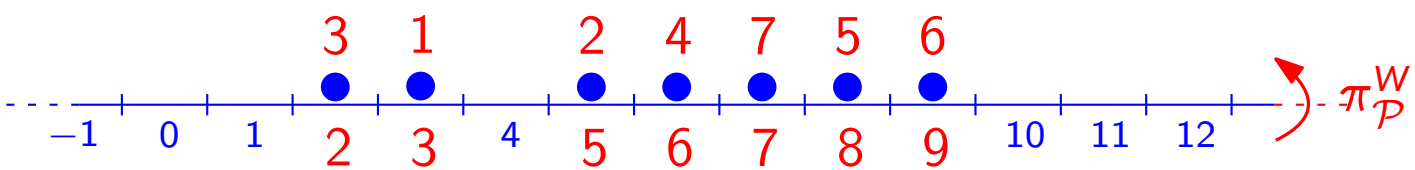

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Then $\mathrm{lastSpot}_{\mathcal{P}}(Wa) \in \{t-1, u+1\}$. $(= \{\mathit{Left}, \mathit{Right}\})$

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Definition. A parking procedure is a function $\mathcal{P} : \mathbb{Z}^* \to \text{Finset}(\mathbb{Z})$ satisfying the three conditions (Accessible parking),(Lucky parking),(Close parking).

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Remark (1). A $\mathcal{P}$-parking word of length r necessarily has all letters $a_i \in \{1, \dots, r\}$.

Remark (2). Permutations of $\{1, \dots, r\}$ are $\mathcal{P}$-parking for any procedure $\mathcal{P}$.

The procedure $\mathcal{P}_{CS}$

To characterize a parking procedure $\mathcal{P}$, we simply need a rule to pick Left or Right when our desired spot is already occupied.

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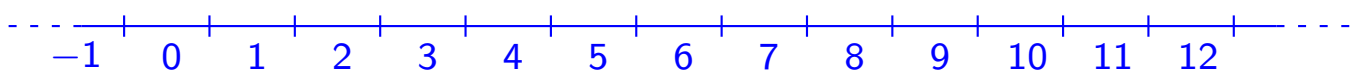

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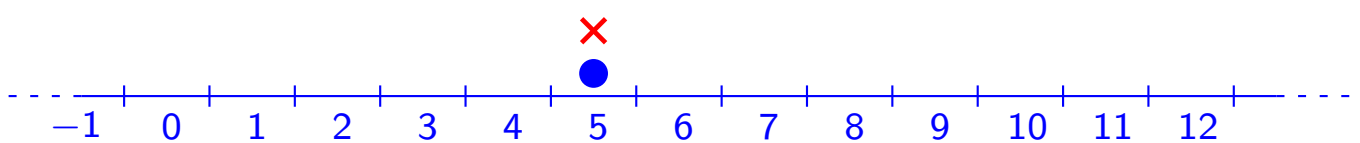

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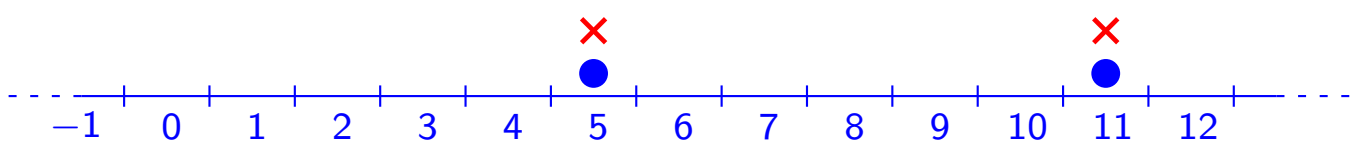

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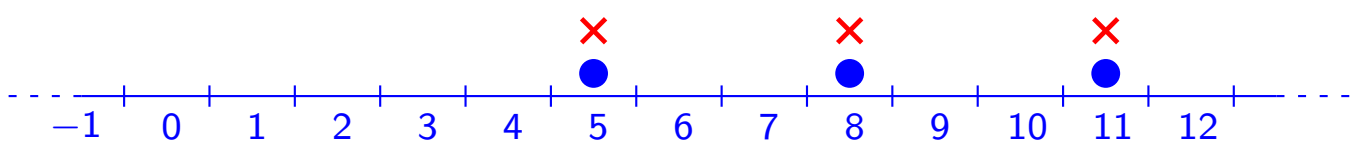

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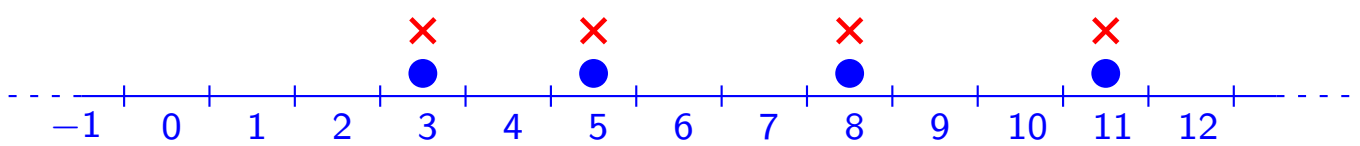

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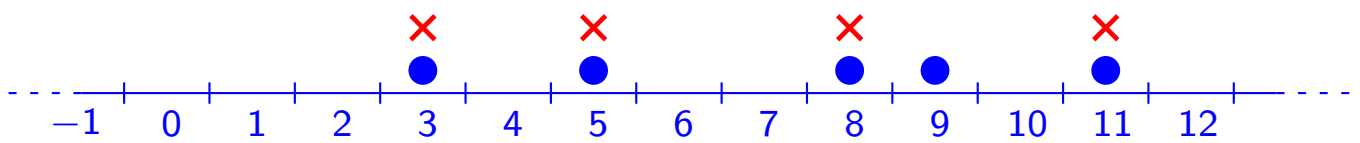

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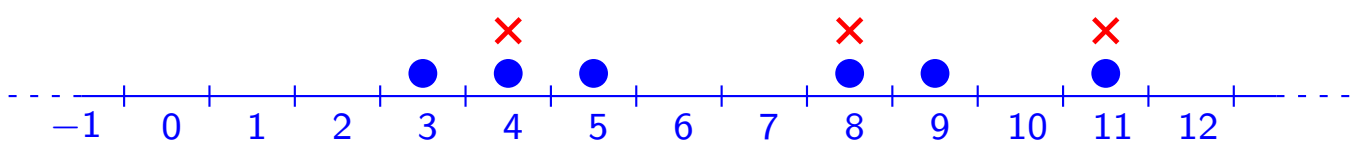

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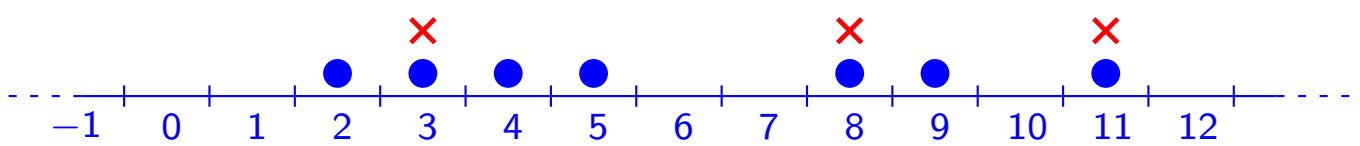

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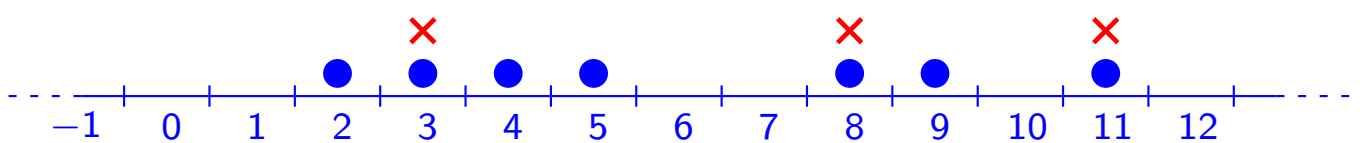

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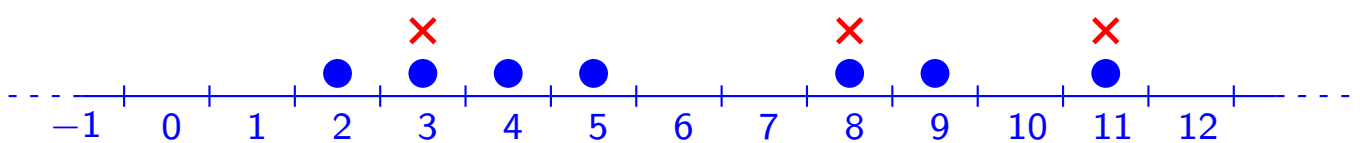

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Definition (Markov property). A procedure $\mathcal{P}$ is called *Markovian* if there is a function $M : \text{Finset}(\mathbb{Z}) \times \mathbb{Z} \to \text{Fin}(\mathbb{Z})$ such that $\mathcal{P}(a_1 \cdots a_r a) = M(\mathcal{P}(a_1 \cdots a_r), a)$

$\mathcal{P}_{CS}$ is not Markovian, while $\mathcal{P}_{classical}$ clearly is.

Encoding as binary forests

(**Goal**: Lift a parking procedure $\mathcal{P}$ to an injective encoding $\widehat{\mathcal{P}}$.)

Definition. A binary forest F with support $A \in \mathrm{Finset}(\mathbb{Z})$ is the data of a binary tree with $|I|$ nodes for each interval I of A.

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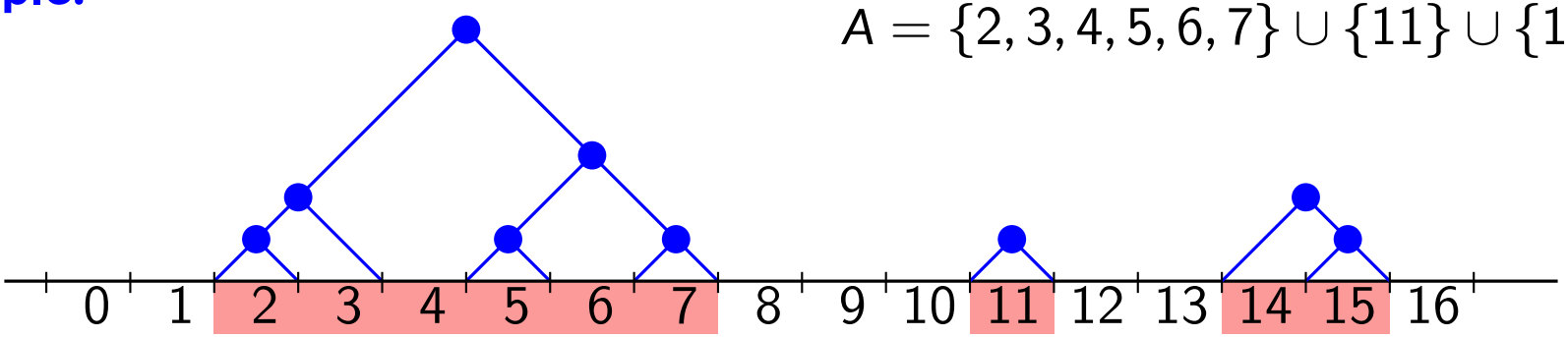
$$A = \{2,3,4,5,6,7\} \cup \{11\} \cup \{14,15\}$$



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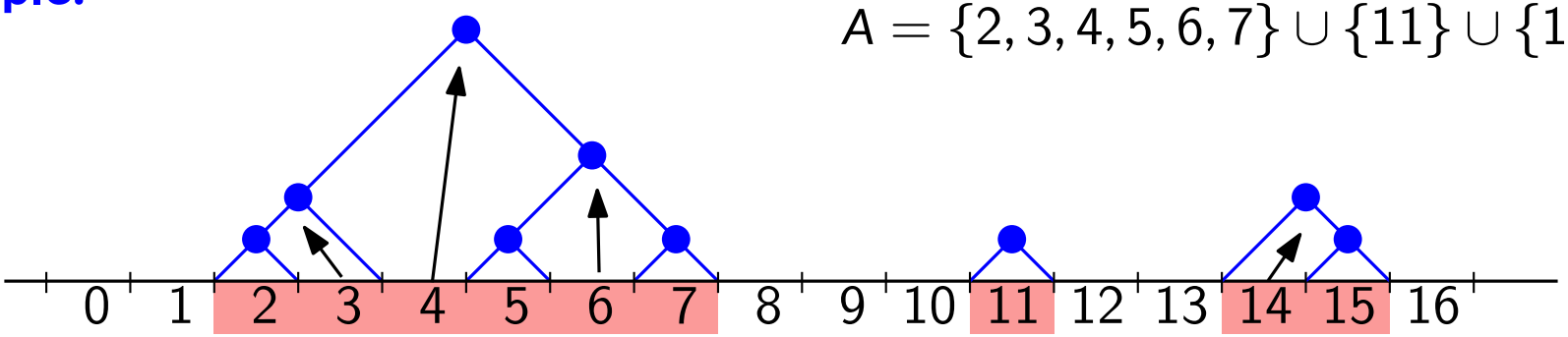

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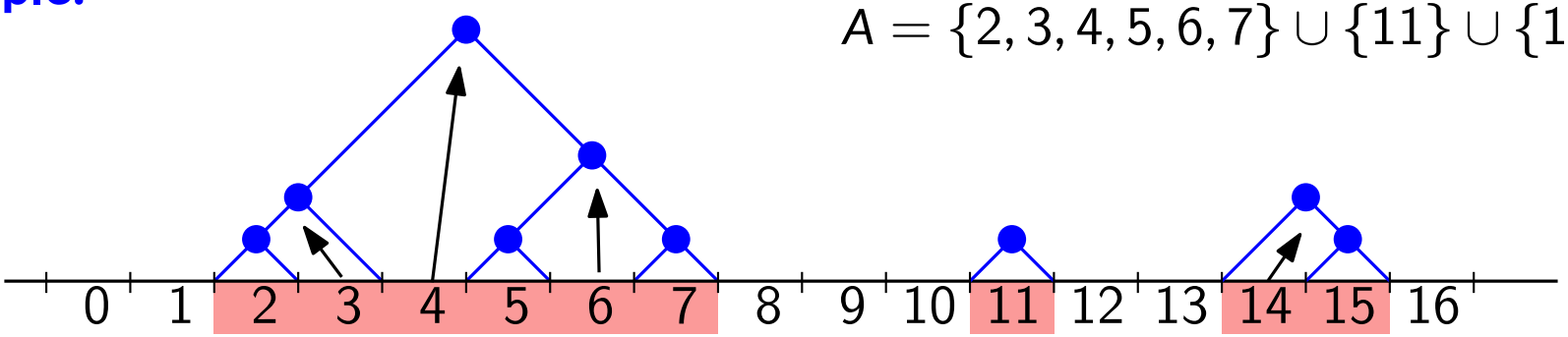

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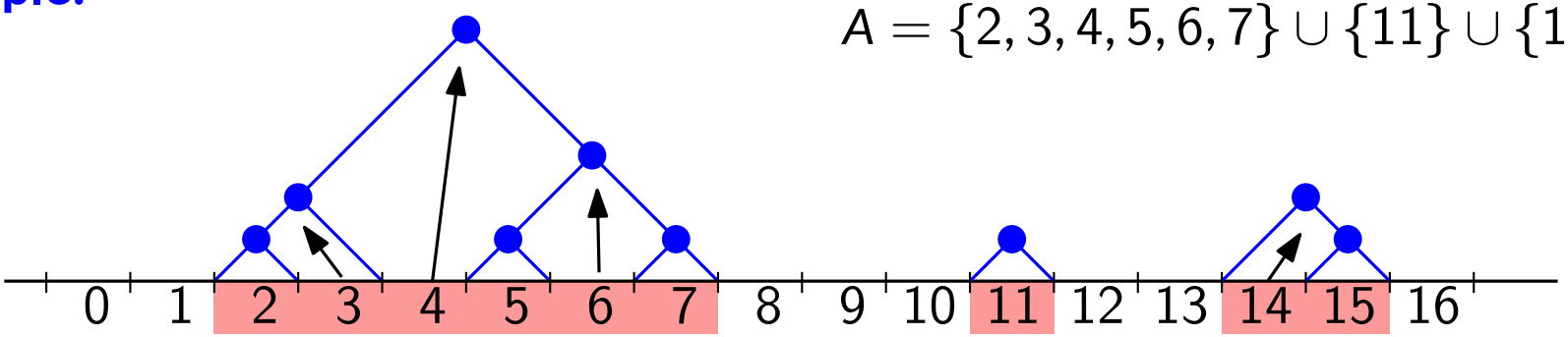

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Labelings of the forest F

Information about the original desired spots

Decreasing forest, encodes exactly the bijection $\pi_{\mathcal{P}}^{W}$

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Definition (by example). $W = a_1a_2a_3a_4a_5a_6a_7 = 5.11.8.3.8.4.3$

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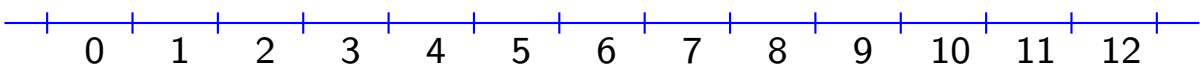

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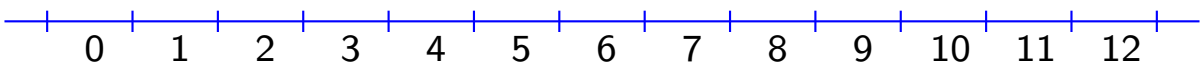

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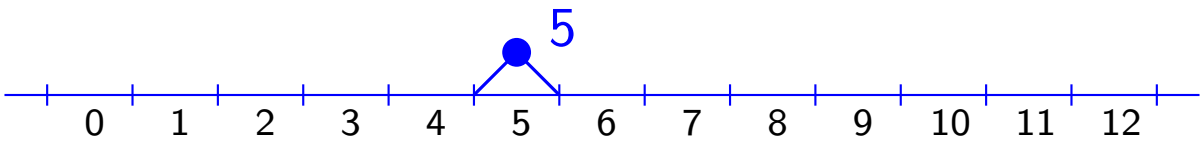

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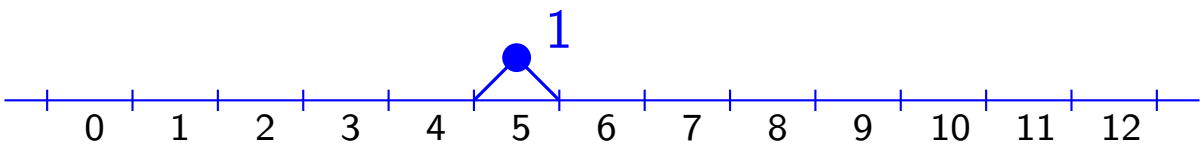

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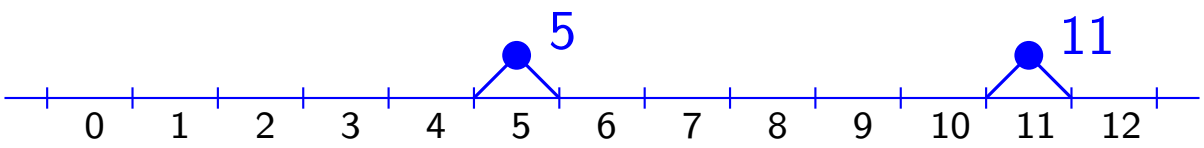

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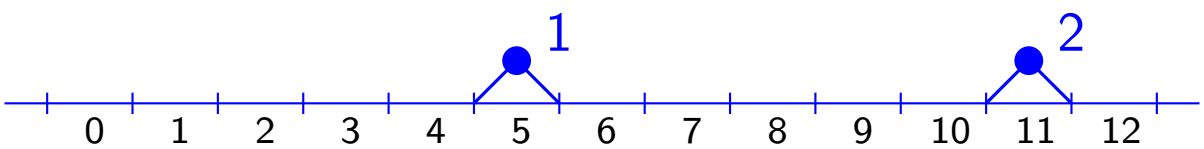

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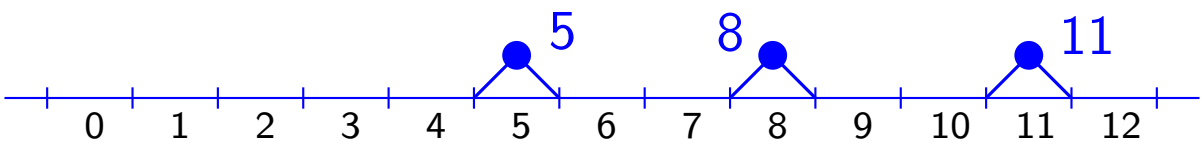

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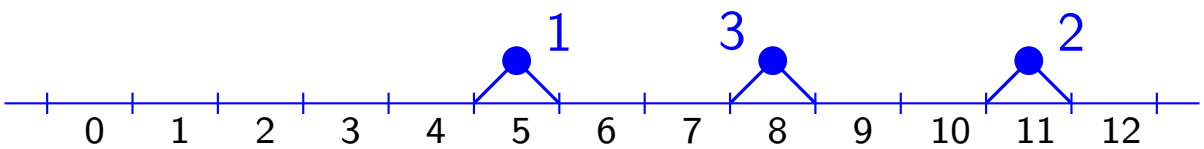

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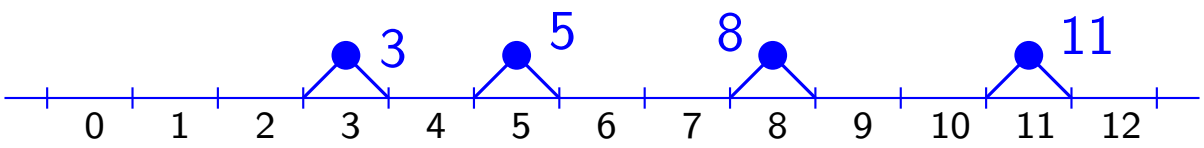

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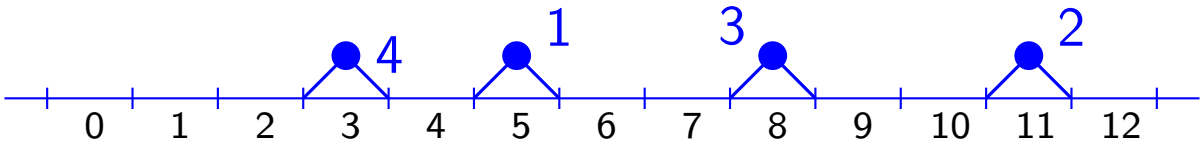

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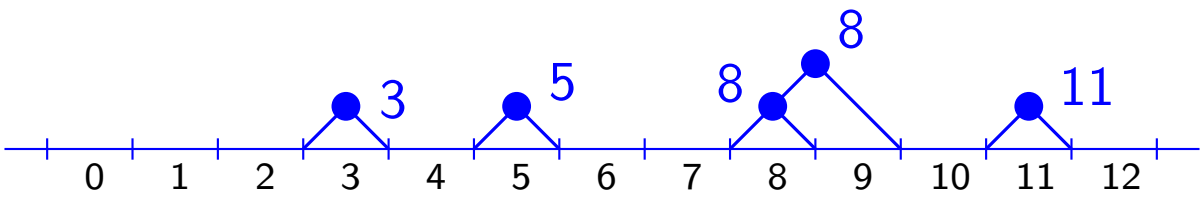

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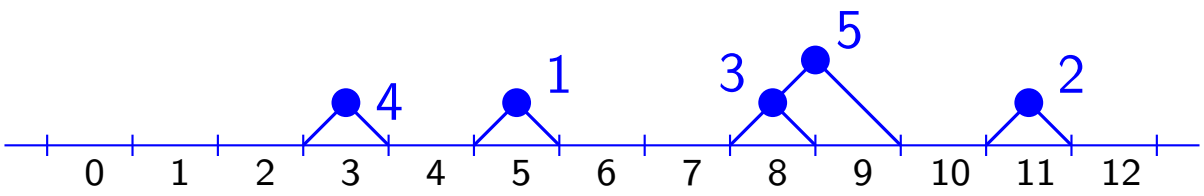

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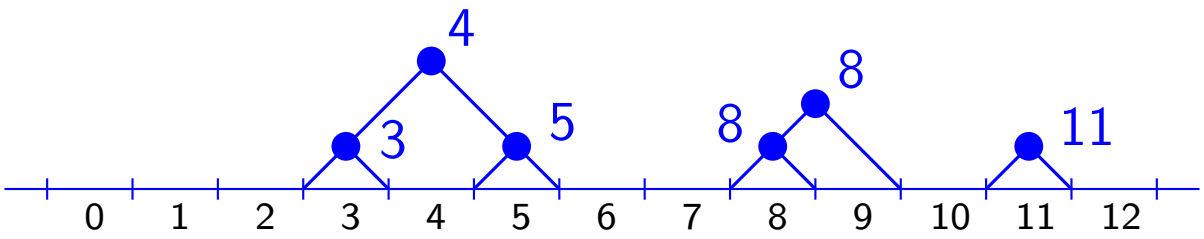

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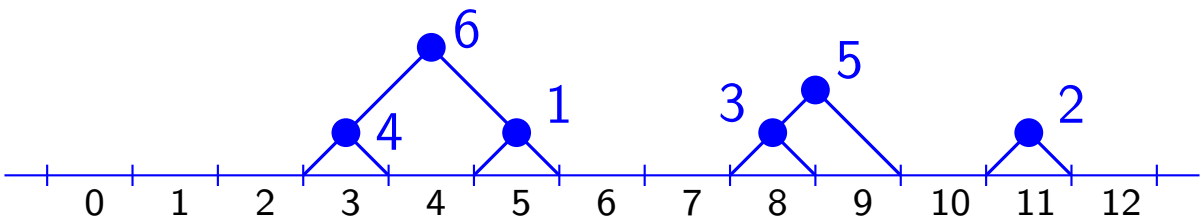

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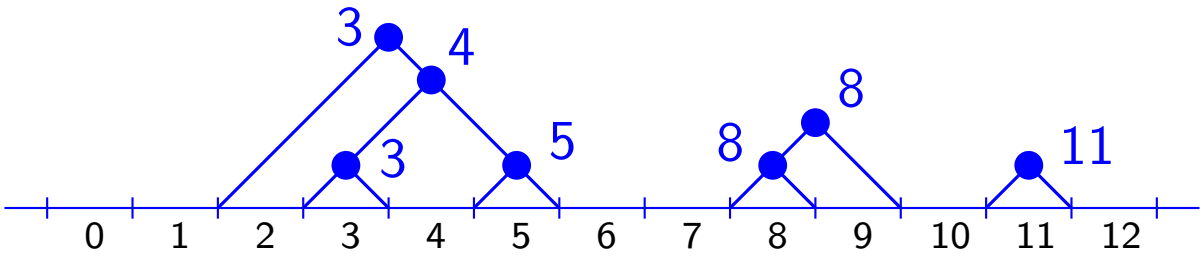

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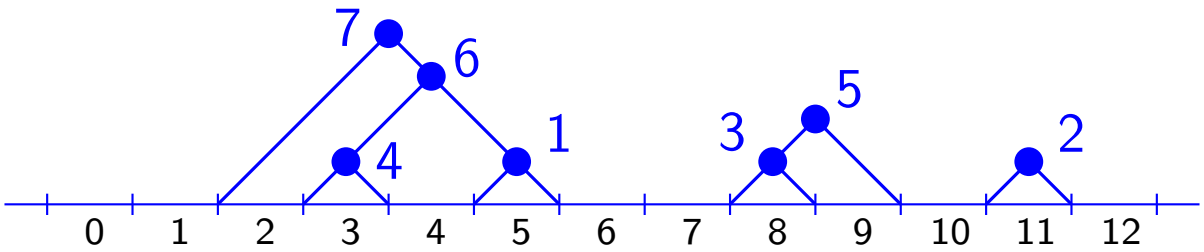

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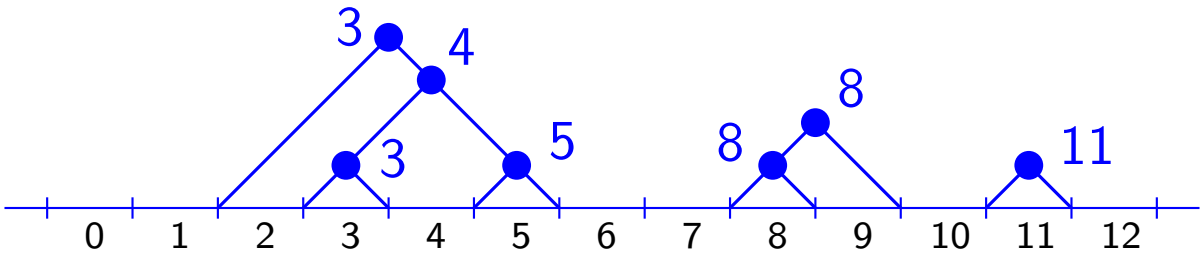

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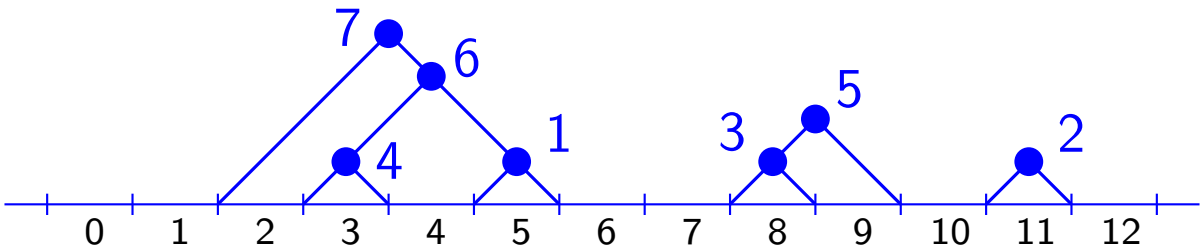

Definition of $\widehat{\mathcal{P}}$. The forest F is constructed inductively: if the actual parking spot is j, create a root node with canonical label j, and attach subtrees if necessary.
If the desired spot was a_i, the label in P is a_i and the label in Q is i.

$$\widehat{\mathcal{P}}(W) = (P(W), Q(W)) \in \bigcup_{F \text{ forest}} \mathcal{F}(F) \times \mathrm{Dec}(F).$$

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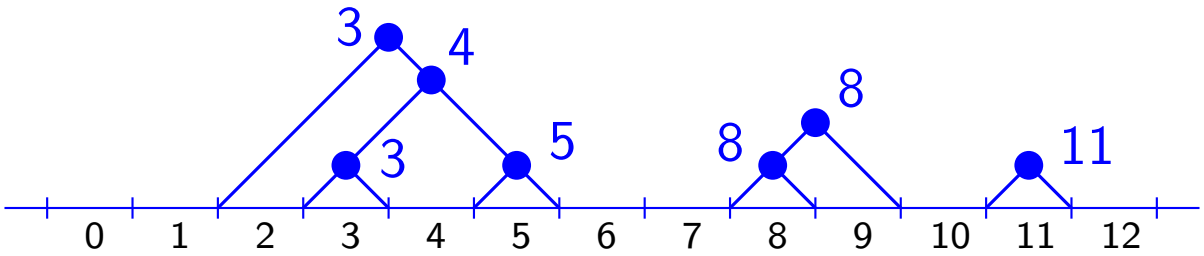
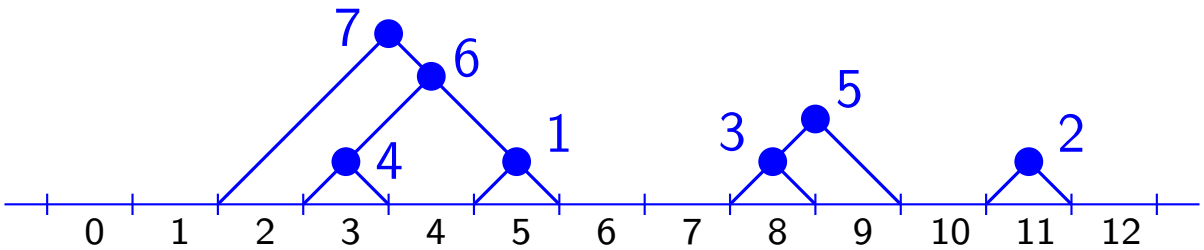


$Q(W)$



Definition of $\widehat{\mathcal{P}}$. The forest F is constructed inductively: if the actual parking spot is j, create a root node with canonical label j, and attach subtrees if necessary.
If the desired spot was a_i, the label in P is a_i and the label in Q is i.

$$\widehat{\mathcal{P}}(W) = (P(W), Q(W)) \in \bigcup_{F \text{ forest}} \mathcal{F}(F) \times \mathrm{Dec}(F).$$

Some facts.

- $\widehat{\mathcal{P}}$ is injective, and $\mathcal{P}(W)$ is the support of the forest F in $\widehat{\mathcal{P}}(W)$.
- If the letters of W are distinct, one has essentially the *Sylvester correspondence*.
- But the image cannot be "nice" for general $\mathcal{P}$...

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Definition. A parking procedure $\mathcal{P}$ is called local if it satisfies the extra axioms (Shift invariance) and (Local decision).

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Proposition. If $\mathcal{P}$ is local, there is a class of labeled forests $\mathcal{F}_{\mathcal{P}}(F) \subset \mathcal{F}(F)$ such that $\widehat{\mathcal{P}} : \mathbb{Z}^* \to \bigcup_{F \text{ forest}} \mathcal{F}_{\mathcal{P}}(F) \times \text{Dec}(F)$ is a bijection.

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Theorem ([N. '22+]). For any local parking procedure $\mathcal{P}$, the number of $\mathcal{P}$-parking words of size r is $(r+1)^{r-1}$.

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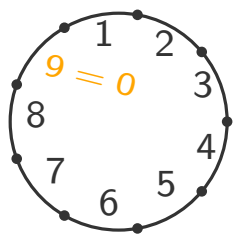

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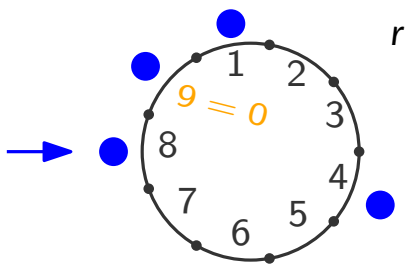

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Example. Suppose $w = \overline{1941}$ has $\mathcal{P}^9(w) = \{\bar{8}, \bar{9} = \bar{0}, \bar{1}\} \sqcup \{\bar{4}\} \subset \mathbb{Z}_9$

Pick $\bar{a} = \bar{8}$. We lift $\{\bar{8}, \bar{9} = \bar{0}, \bar{1}\}$ to $\{1, 2, 3\}$, $\bar{8}$ to 1, and $\overline{1941}$ to $\overline{3136}$.

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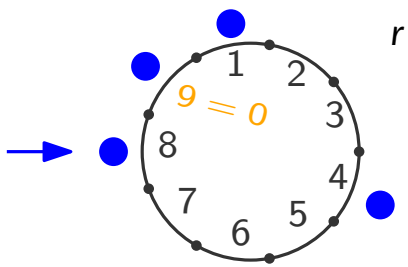

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The theorem then follows from the following lemmas:

Lemma (1). Let $W \in \{1, \dots, r+1\}^r$. Then
W is a $\mathcal{P}$-parking function $\Leftrightarrow \mathcal{P}^{r+1}(W \mod (r+1))$ has empty spot $\bar{0}$.

Lemma (2). Let $w \in \mathbb{Z}_{r+1}^r$ have empty spot k.
Then $\bar{\tau}^i(w) \in \mathbb{Z}_{r+1}^r$ has empty spot $k+i$ for $i = 0, 1, \dots, r$.

□

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Example (2). Fix a parameter $q \in [0,\infty]$. Suppose one wants to park at $i \in [1,n]$. Then go Right with probability $\frac{1+q+\cdots+q^{i-1}}{1+q+\cdots+q^{n-1}}$ (and thus Left with probability $\frac{q^i+\cdots+q^{n-1}}{1+q+\cdots+q^{n-1}}$). This determines a local procedure that is abelian.

This last procedure can be seen a series of random walks, and has many nice properties. (cf. [Diaconis-Fulton, Tewari-N.])

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Let S be any set, and consider the alphabet $A = \mathbb{Z} \times S$. The set S represents some extra information: in terms of cars, one might consider its brand or color.

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