

A branch statistic for trees

Interpreting coefficients of Catalan characteristic polynomials

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June 2022

Joint work with Priyavrat Deshpande

- 1 Hyperplane arrangements
- 2 Catalan arrangements and trees
- 3 Branches of trees
- 4 Other arrangements

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Definition (Hyperplane Arrangement)

A hyperplane arrangement is a finite set \mathcal{A} of affine hyperplanes in \mathbb{R}^n .

Definition (Region)

A region (or chamber) of an arrangement \mathcal{A} is a connected component of

$$\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H.$$

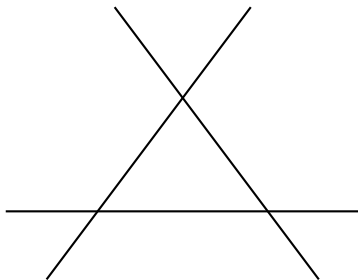
The number of regions is denoted by $r(\mathcal{A})$.

Examples

In \mathbb{R}^1



In \mathbb{R}^2

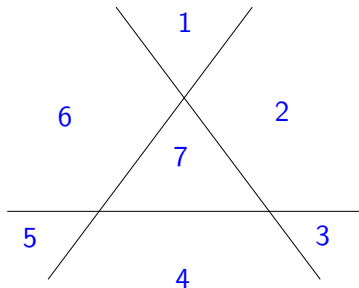


Examples

In \mathbb{R}^1



In \mathbb{R}^2



- Any hyperplane in \mathbb{R}^n is of the form:

$$H = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \dots + a_nx_n = c\}.$$

- Defining polynomial of H :

$$P_H = a_1x_1 + \dots + a_nx_n - c.$$

- Any region of \mathcal{A} is an intersection of half-spaces of the form $P_H > 0$ or $P_H < 0$ for each $H \in \mathcal{A}$.

Characteristic polynomial

- $L_{\mathcal{A}}$ is the poset of non-empty intersections of hyperplanes in \mathcal{A} ordered by reverse inclusion.
- The möbius function $\mu : L_{\mathcal{A}} \rightarrow \mathbb{Z}$ is defined as:

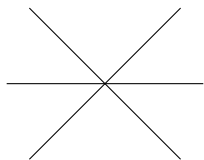
$$\mu(x) = \begin{cases} 1, & \text{if } x = \mathbb{R}^n \\ -\sum_{y < x} \mu(y), & \text{otherwise.} \end{cases}$$

- The characteristic polynomial of \mathcal{A} is defined as:

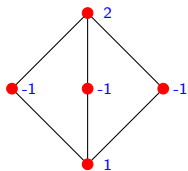
$$\chi_{\mathcal{A}}(t) = \sum_{x \in L_{\mathcal{A}}} \mu(x) t^{\dim(x)}.$$

Examples

\mathcal{A}

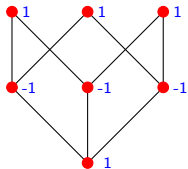
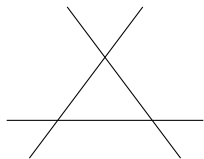


$L_{\mathcal{A}}$



$\chi_{\mathcal{A}}(t)$

$$t^2 - 3t + 2$$



$$t^2 - 3t + 3$$

Property of $\chi_{\mathcal{A}}(t)$

Theorem

For any arrangement \mathcal{A} in \mathbb{R}^n , the characteristic polynomial is of the form

$$\chi_{\mathcal{A}}(t) = \sum_{i=0}^n (-1)^{n-i} c_i t^i$$

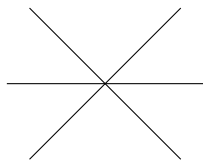
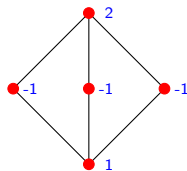
where $c_i \in \mathbb{Z}_{\geq 0}$ for all $0 \leq i \leq n$.

Theorem (Zaslavsky, 1975)

Let \mathcal{A} be an arrangement in \mathbb{R}^n . Then,

$$\begin{aligned}r(\mathcal{A}) &= (-1)^n \chi_{\mathcal{A}}(-1) \\ &= \sum_{i=0}^n c_i.\end{aligned}$$

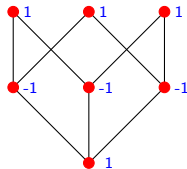
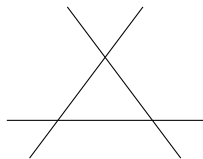
Examples

 \mathcal{A}  $L_{\mathcal{A}}$  $\chi_{\mathcal{A}}(t)$

$$t^2 - 3t + 2$$

 $r(\mathcal{A})$

6



$$t^2 - 3t + 3$$

7

Interpreting the coefficients of $\chi_{\mathcal{A}}(t)$

- Let \mathcal{A} be an arrangement in \mathbb{R}^n and $\chi_{\mathcal{A}}(t) = \sum_{i=0}^n (-1)^{n-i} c_i t^i$.
- By Zaslavsky's theorem, $r(\mathcal{A}) = \sum_{i=0}^n c_i$.

Question

When the regions of an arrangement are in bijection with a combinatorially defined set, is there a *statistic* on the set that induces this breakup?

Definition (Statistic)

A statistic on a set of combinatorial objects \mathcal{O} is a map $f : \mathcal{O} \rightarrow \mathbb{Z}_{\geq 0}$.

Definition (Distribution)

The distribution induced by a statistic f on \mathcal{O} is the sequence given by

$$c_i = |f^{-1}(i)| \quad \text{for all } i \in \mathbb{Z}_{\geq 0}.$$

In particular, we have

$$|\mathcal{O}| = \sum_{i \geq 0} c_i.$$

Braid arrangement

- $\mathcal{A}_n = \{x_i = x_j \mid 1 \leq i < j \leq n\}$.
- Regions $\leftrightarrow \mathfrak{S}_n$.
- $\chi_{\mathcal{A}_n}(t) = \sum_{i=0}^n s(n, i)t^i$.
- Coefficients count permutations according to number of cycles.

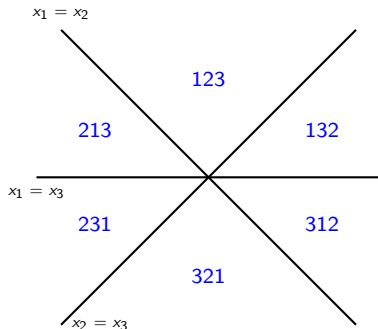


Figure: \mathcal{A}_3 on $x_1 + x_2 + x_3 = 0$.

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Catalan arrangement

Arrangement \mathcal{C}_n in \mathbb{R}^n defined as:

$$\{x_i = x_j, x_i = x_j + 1 \mid i, j \in [n], i \neq j\}.$$

Specifying a region is same as specifying a **valid** total order on

$$x_1, \dots, x_n, x_1 + 1, \dots, x_n + 1.$$

Example

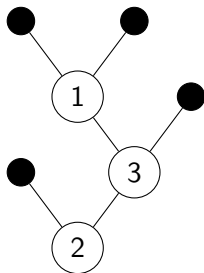
$$x_2 < x_1 < x_2 + 1 < x_1 + 1 < x_4 < x_4 + 1 < x_3 < x_3 + 1$$

represents a region of \mathcal{C}_4 .

Valid: • $x_i < x_i + 1$ • $x_i < x_j \Rightarrow x_i + 1 < x_j + 1$.

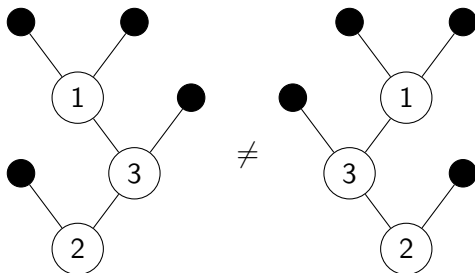
Labeled binary trees

A binary tree with n nodes labeled distinctly using $[n]$.



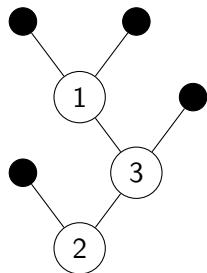
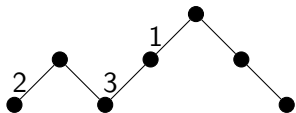
Labeled binary trees

A binary tree with n nodes labeled distinctly using $[n]$.



Regions \leftrightarrow trees

$$x_2 < x_2 + 1 < x_3 < x_1 < x_3 + 1 < x_1 + 1$$



Arrangement $\mathcal{C}_n^{(m)}$ in \mathbb{R}^n defined as:

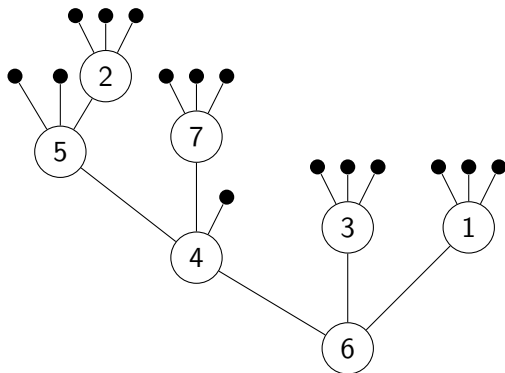
$$\{x_i = x_j + k \mid i, j \in [n], i \neq j, k \in [0, m]\}.$$

Regions: Valid total orders on

$$x_1, \dots, x_n, x_1 + 1, \dots, x_n + 1, \dots, x_1 + m, \dots, x_n + m.$$

Labeled $(m + 1)$ -ary trees

An $(m + 1)$ -ary tree with n nodes labeled distinctly using $[n]$.



Characteristic polynomial and region count

Theorem (Athanasiadis, 1996)

The characteristic polynomial of $\mathcal{C}_n^{(m)}$ is given by

$$\chi_{\mathcal{C}_n^{(m)}}(t) = t(t - mn - 1)(t - mn - 2) \cdots (t - mn - (n - 1)).$$

Corollary

The number of regions of $\mathcal{C}_n^{(m)}$ is given by

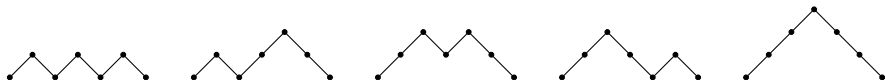
$$r(\mathcal{C}_n^{(m)}) = \frac{n!}{mn + 1} \binom{(m + 1)n}{n}.$$

Aside: Fuss-Catalan numbers

Definition (n^{th} m -Catalan number)

The number of unlabeled $(m + 1)$ -ary trees with n nodes given by

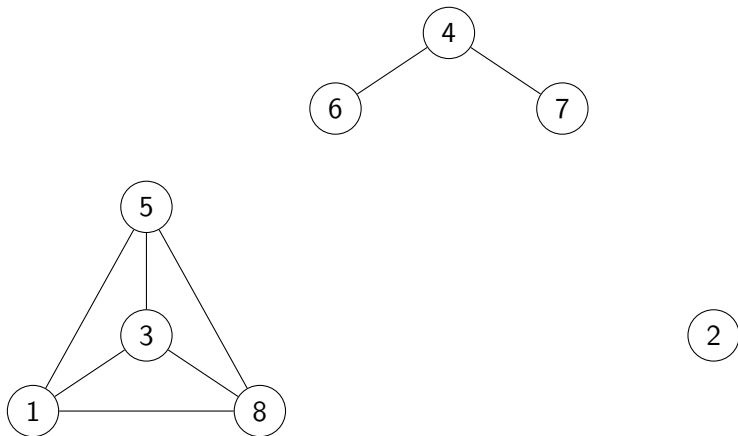
$$\frac{1}{mn + 1} \binom{(m + 1)n}{n}.$$



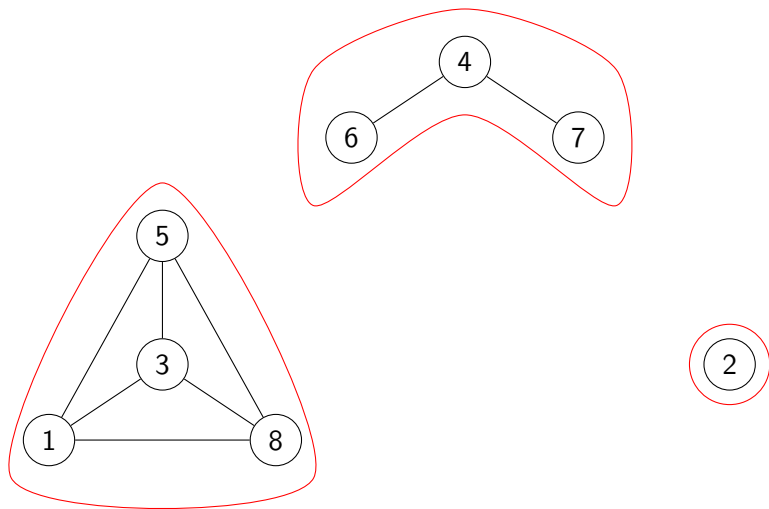
Many other interpretations. Over 200 in the case $m = 1$!

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Exponential structures: Graphs



Connected components



Graph structure on vertex set V



Partition of V : $\{B_1, B_2, \dots, B_k\}$



Connected graph structure on each B_i .

Exponential structures

- For any finite set V , $S(V)$ is the set of S -structures on V .
- $S_c(V) \subseteq S(V)$ is the set of “connected” S -structures on V .

S -structure on label set V



Partition of V : $\{B_1, B_2, \dots, B_k\}$

+

Connected S -structure on each B_i .

Counting structures

For any $n, k \geq 0$,

$s(n)$ = Number of S -structures on $[n]$

$s_k(n)$ = Number of S -structures on $[n]$ with k “components”

Theorem (Stanely, EC2)

We have the equality of formal power series:

$$\sum_{n,k \geq 0} s_k(n) t^k \frac{x^n}{n!} = \left(\sum_{n \geq 0} s(n) \frac{x^n}{n!} \right)^t.$$

Find a statistic on the labeled $(m + 1)$ -ary trees with n nodes whose distribution is given by the coefficients of $\chi_{\mathcal{C}_n}^{(m)}$.

Characteristic polynomial and region count

Recall

For any $m, n \geq 1$,

- $\chi_{\mathcal{C}_n^{(m)}}(t) = t(t - mn - 1)(t - mn - 2) \cdots (t - mn - (n - 1))$.
- $r(\mathcal{C}_n^{(m)}) = \frac{n!}{mn+1} \binom{(m+1)n}{n}$.

Theorem (Stanley, 1995)

For any $m \geq 1$, we have

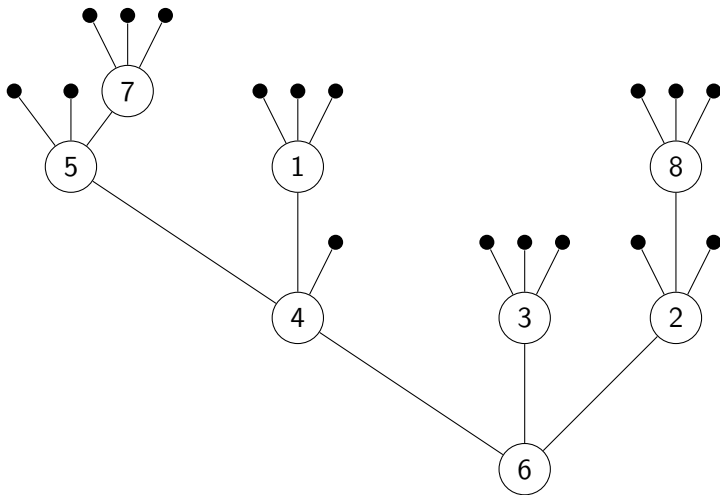
$$\sum_{n \geq 0} \chi_{\mathcal{C}_n^{(m)}}(t) \frac{x^n}{n!} = \left(\sum_{n \geq 0} (-1)^n r(\mathcal{C}_n^{(m)}) \frac{x^n}{n!} \right)^{-t}.$$

- $C(m, n, k) =$ absolute value of the coefficient of t^k in $\chi_{\mathcal{C}_n^{(m)}}(t)$.
- We have

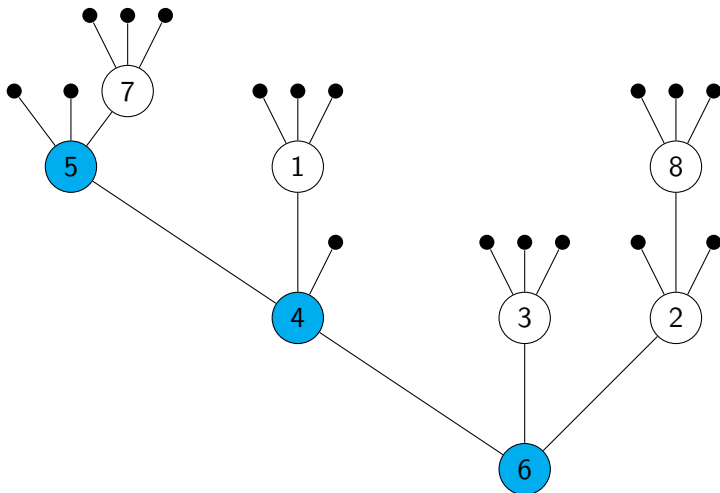
$$\sum_{n, k \geq 0} C(m, n, k) t^k \frac{x^n}{n!} = \left(\sum_{n \geq 0} r(\mathcal{C}_n^{(m)}) \frac{x^n}{n!} \right)^t.$$

- Show $(m + 1)$ -ary tree structure is exponential, then $C(m, n, k)$ is number of trees with k components.

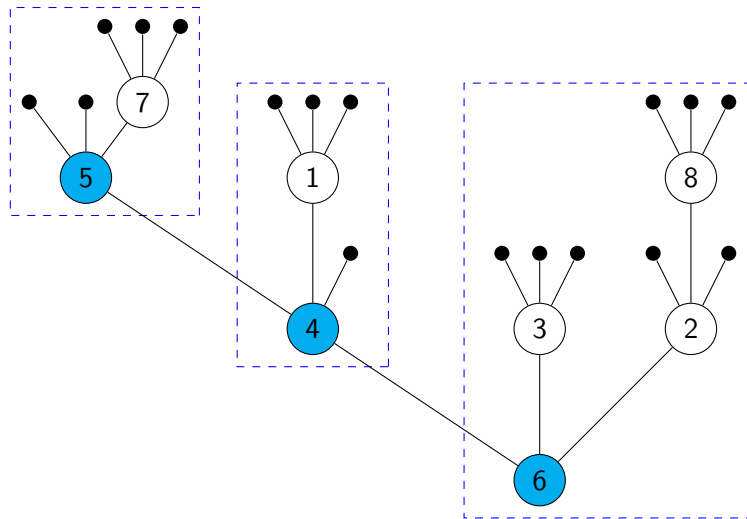
Parts of a tree



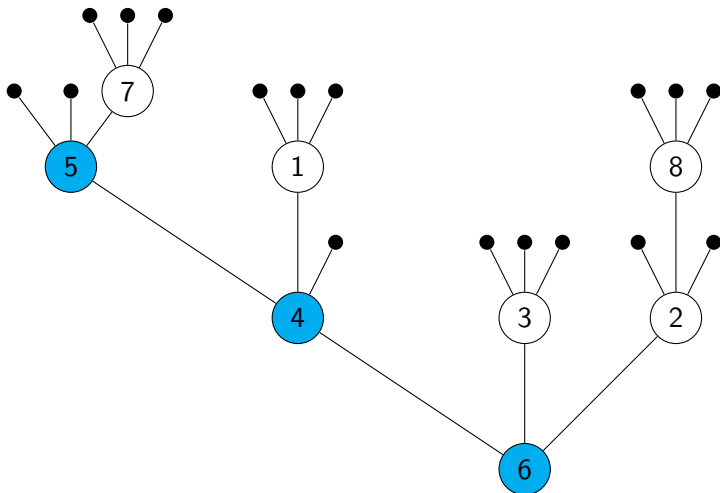
Parts of a tree: Trunk



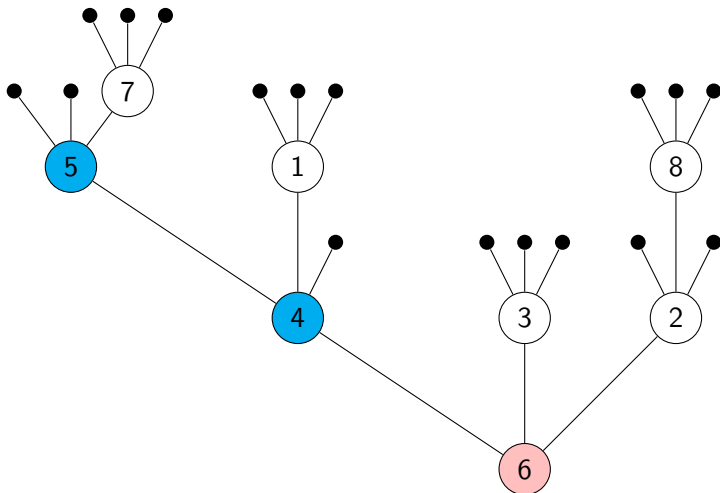
Parts of a tree: Twigs



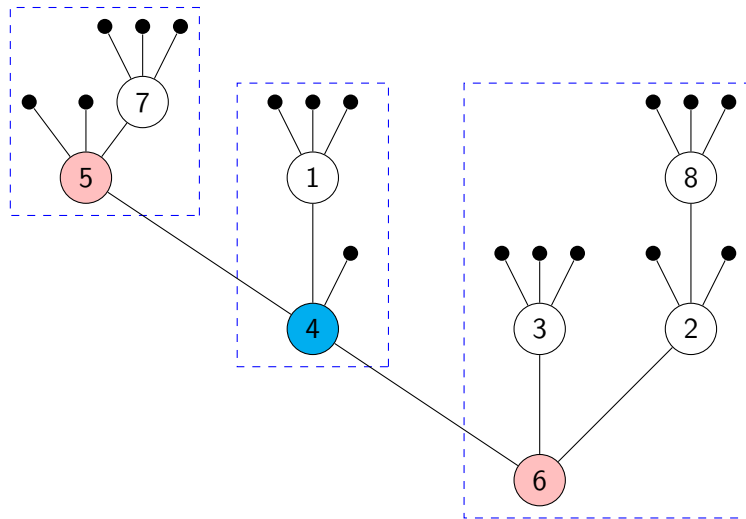
Parts of a tree: Branch nodes



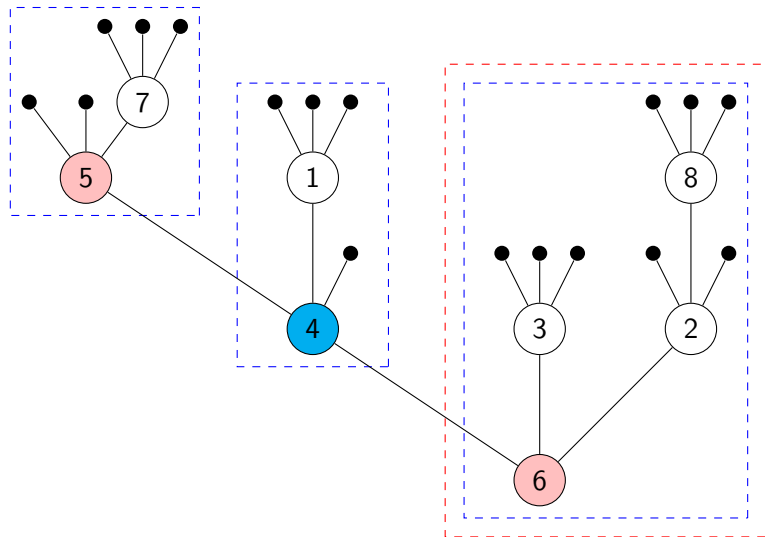
Parts of a tree: Branch nodes



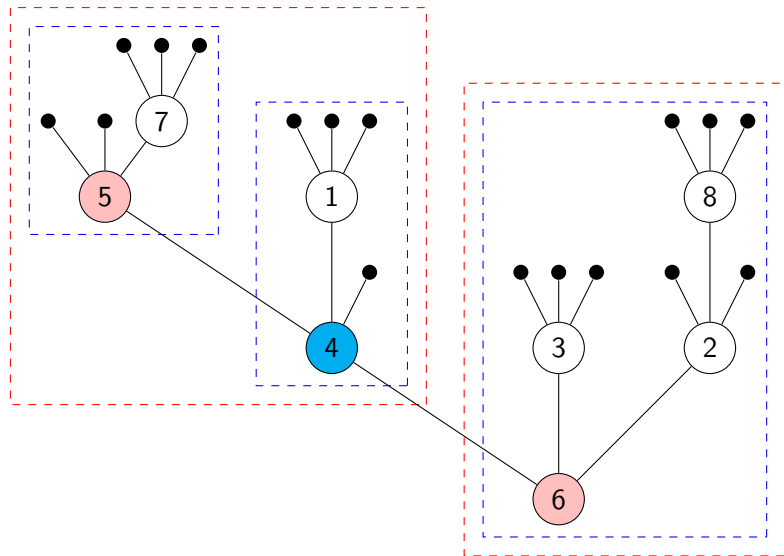
Parts of a tree: Branches



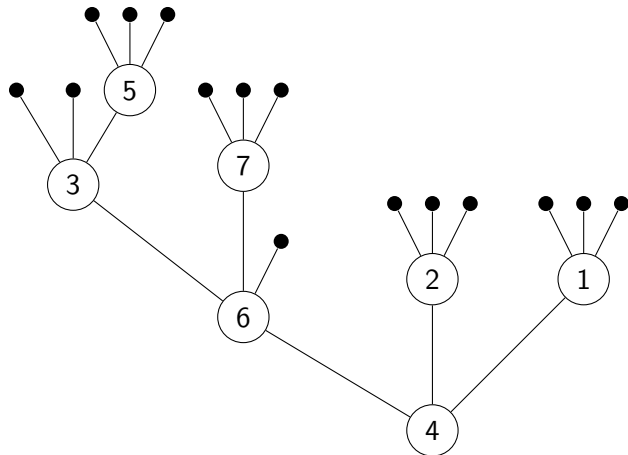
Parts of a tree: Branches



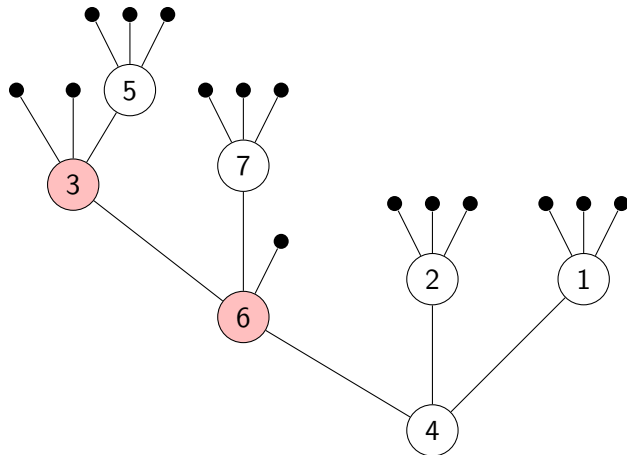
Parts of a tree: Branches



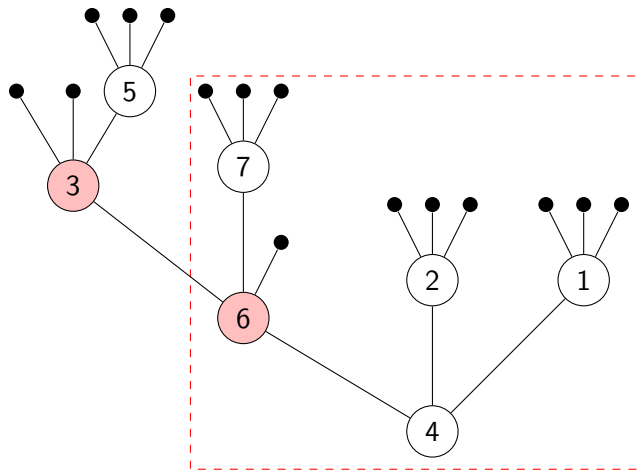
Example 2



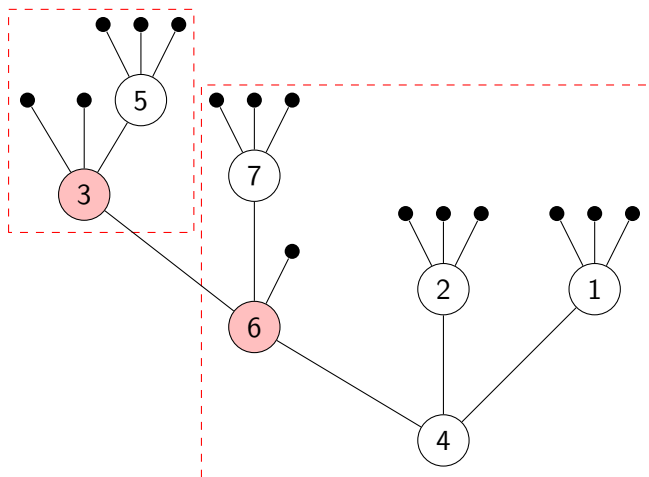
Example 2



Example 2



Example 2



Labeled $(m + 1)$ -ary trees

Definition

Let V be a finite set of positive integers. $\mathcal{T}^{(m)}(V)$ is the set of $(m + 1)$ -ary trees with $|V|$ nodes labeled distinctly using V .

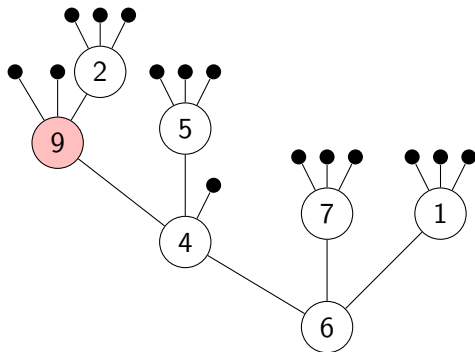
Trees in $\mathcal{T}^{(m)}(V)$ break up into branches just as before.

Connected $(m + 1)$ -ary trees

Definition

Connected trees in $\mathcal{T}^{(m)}(V)$ are those where

- there is only one branch, or equivalently
- last node on the trunk is the largest.

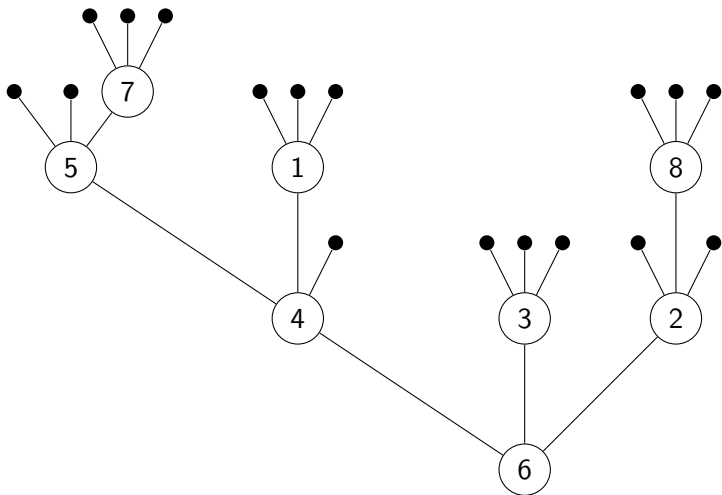


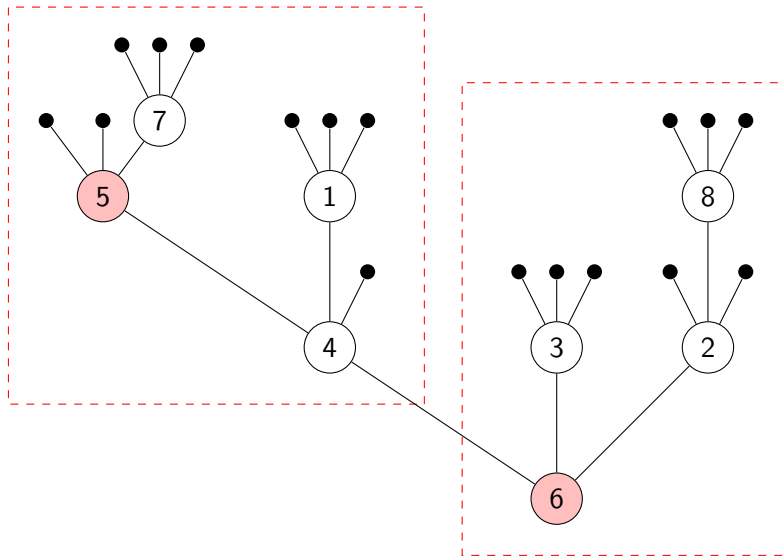
Theorem

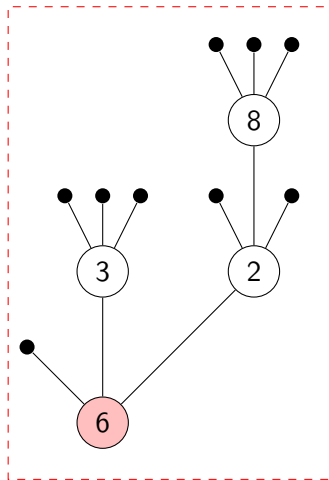
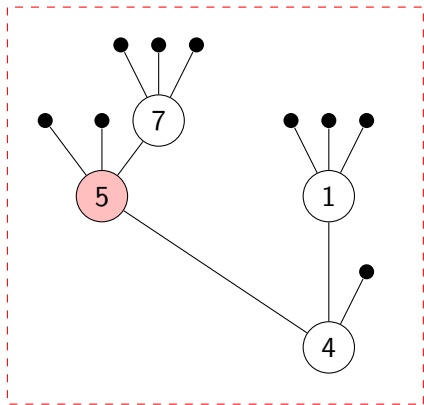
$C(m, n, k)$ is the number of trees in $\mathcal{T}^{(m)}([n])$ with k branches.

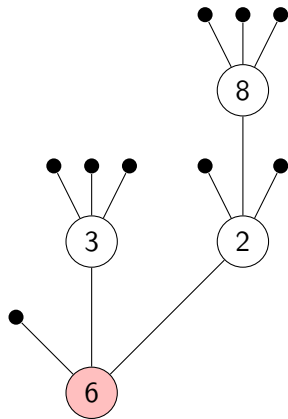
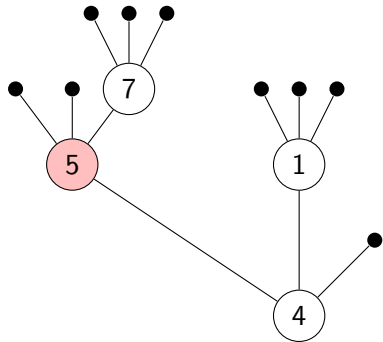
To prove:

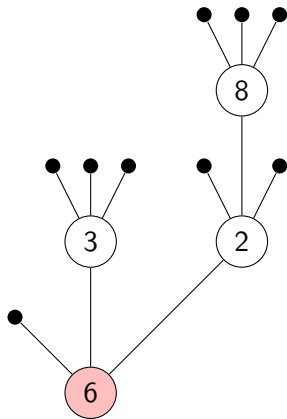
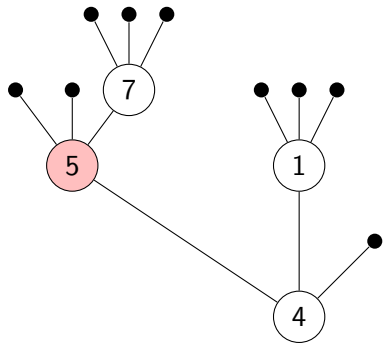
- The branches break a tree into connected trees.
- A tree can be reconstructed from its branches.



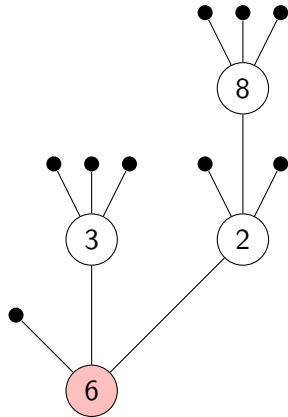
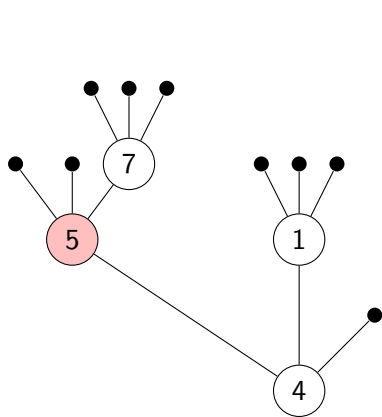




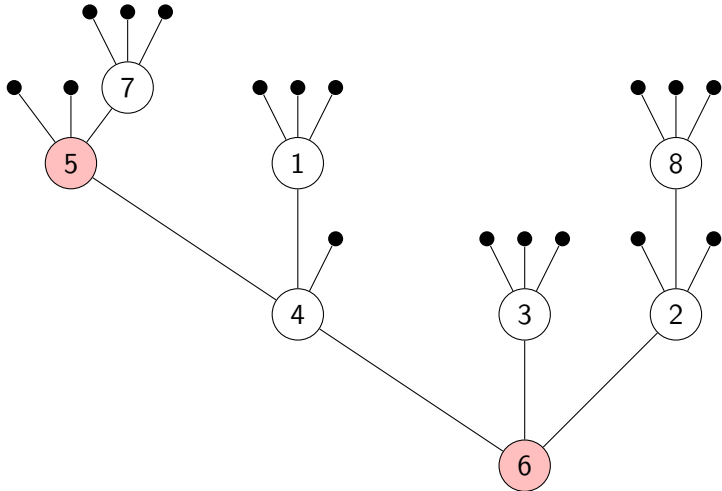




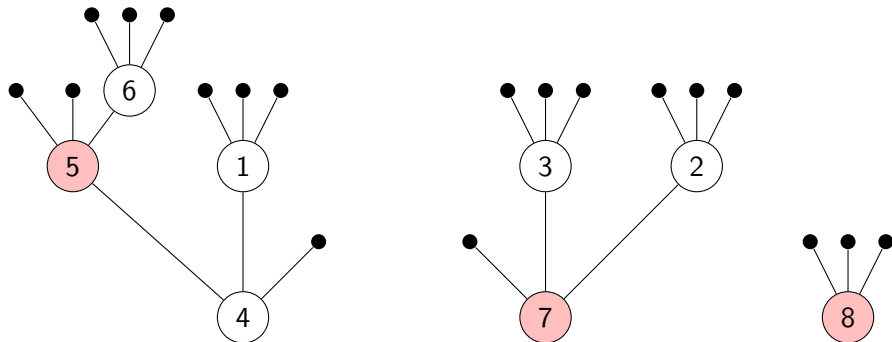
Which first?



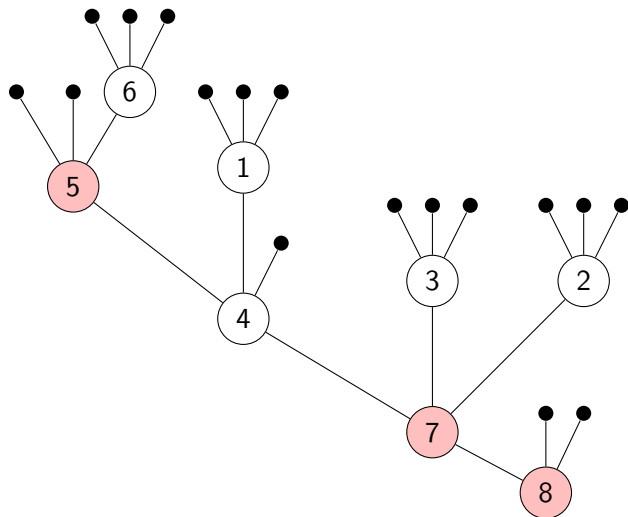
Which first? The one which has the largest trunk label.



Example 2



Example 2



- The m -Catalan arrangement in \mathbb{R}^n is

$$\mathcal{C}_n^{(m)} := \{x_i = x_j + k \mid i, j \in [n], i \neq j, k \in [0, m]\}.$$

- The regions of $\mathcal{C}_n^{(m)}$ correspond to the trees in $\mathcal{T}^{(m)}([n])$.
- The absolute value of the coefficient of t^k in $\chi_{\mathcal{C}_n^{(m)}}(t)$ counts the trees in $\mathcal{T}^{(m)}([n])$ with k branch nodes.

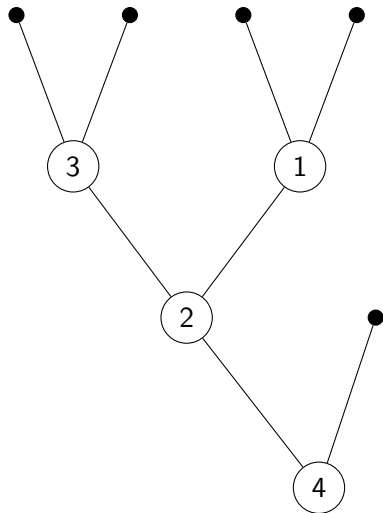
Recall

$C(m, n, k)$ is the number of trees in $\mathcal{T}^{(m)}([n])$ with k branches.

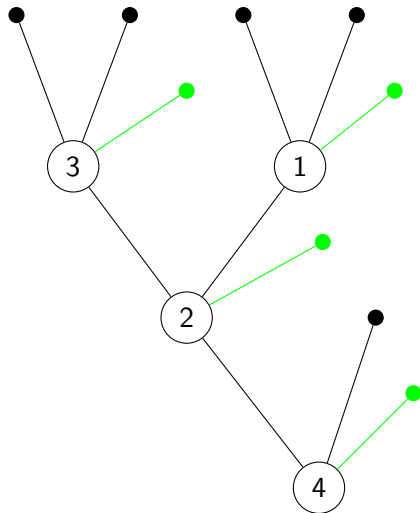
Result

- 1 $C(m, n, k) \leq C(m + 1, n, k)$
- 2 $C(m, n, k) \leq C(m, n + 1, k)$
- 3 $C(m, n, k) \geq C(m, n, k + 1)$

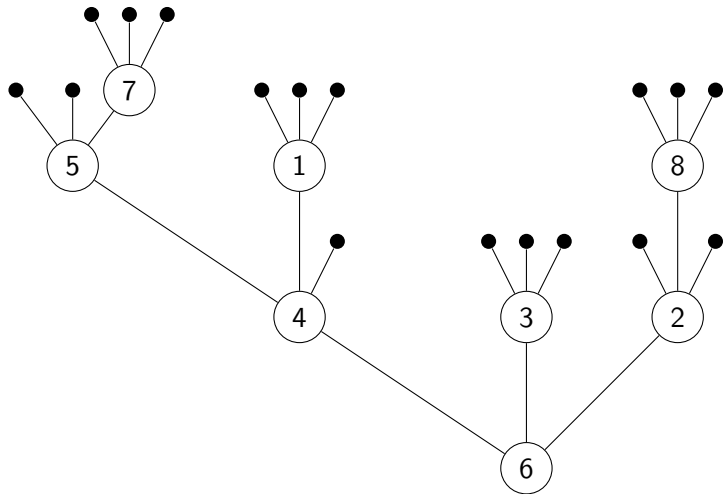
1. $C(m, n, k) \leq C(m + 1, n, k)$



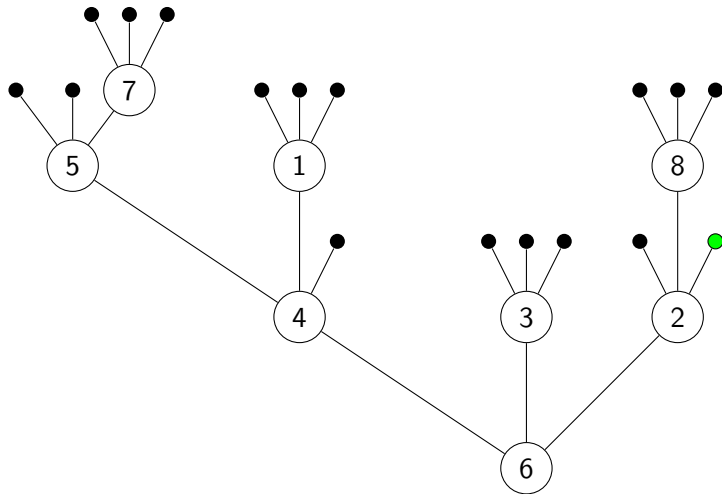
1. $C(m, n, k) \leq C(m + 1, n, k)$



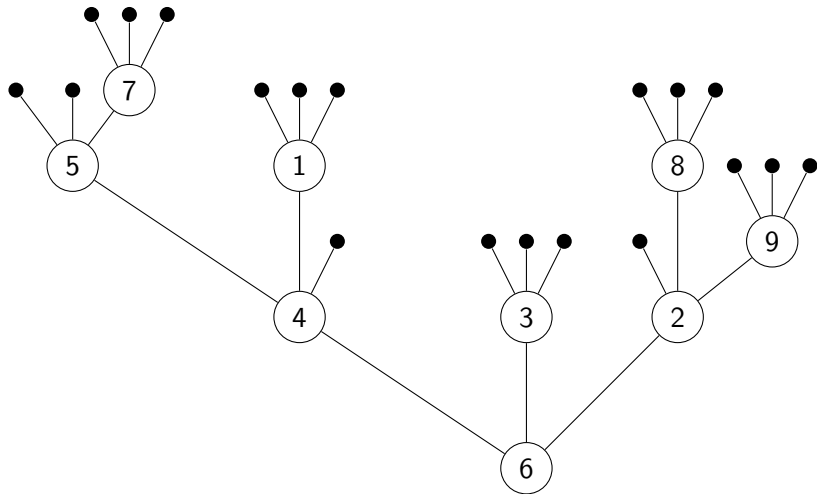
2. $C(m, n, k) \leq C(m, n + 1, k)$



2. $C(m, n, k) \leq C(m, n + 1, k)$



2. $C(m, n, k) \leq C(m, n + 1, k)$



Theorem

There are more connected trees in $\mathcal{T}^{(m)}([n])$ than disconnected trees.

- More trees with only one branch than those with more than one.
- Equivalent to $C(m, n, 1) \geq \sum_{k=2}^n C(m, n, k)$.
- Implies $C(m, n, 1) \geq C(m, n, 2) \geq C(m, n, 3) \geq \dots$.

Theorem

More graphs are connected than disconnected for any vertex set V .



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Stanley's question

(26) [5–] Let

$$\chi_{\mathcal{T}_n}(t) = t^n - a_{n-1}t^{n-1} + \cdots + (-1)^n a_0.$$

For instance,

$$\chi_{\mathcal{T}_3}(t) = t^3 - 3t^2 + 3t - 1$$

$$\chi_{\mathcal{T}_4}(t) = t^4 - 6t^3 + 15t^2 - 17t + 7$$

$$\chi_{\mathcal{T}_5}(t) = t^5 - 10t^4 + 45t^3 - 105t^2 + 120t - 51.$$

By Exercise 25(a), $a_0 + a_1 + \cdots + a_{n-1} + 1$ is the number of threshold graphs on the vertex set $[n]$. Give a combinatorial interpretation of the numbers a_i as the number of threshold graphs with a certain property.

Figure: Exercise 5.26 of Stanley's notes.

Theorem (Deshpande, M., Singh)

The coefficient a_k is the number of threshold graphs with k "odd anchors".

Definition

Arrangements whose hyperplanes are parallel to some braid hyperplane, *i.e.*, of the form

$$x_i = x_j + k \text{ for some } k \in \mathbb{Z}.$$

DEFORMATIONS OF THE BRAID ARRANGEMENT AND TREES

DEDICATED TO IRA GESSEL FOR HIS RETIREMENT

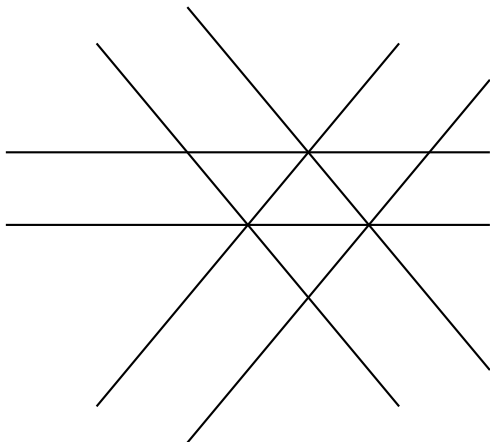
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ABSTRACT. We establish general counting formulas and bijections for deformations of the braid arrangement. Precisely, we consider real hyperplane arrangements such that all the hyperplanes are of the form $x_i - x_j = s$ for some integer s . Classical examples include the braid, Catalan, Shi, semiorder and Linial arrangements, as well as graphical arrangements. We express the number of regions of any such arrangement as a signed count of decorated plane trees. The characteristic and coboundary polynomials of these arrangements also have simple expressions in terms of these trees.

We then focus on certain “well-behaved” deformations of the braid arrangement that we call *transitive*. This includes the Catalan, Shi, semiorder and Linial arrangements, as well as many other arrangements appearing in the literature. For any transitive deformation of the braid arrangement we establish a simple bijection between regions of the arrangement and a set of labeled plane trees defined by local conditions. This answers a question of Gessel.

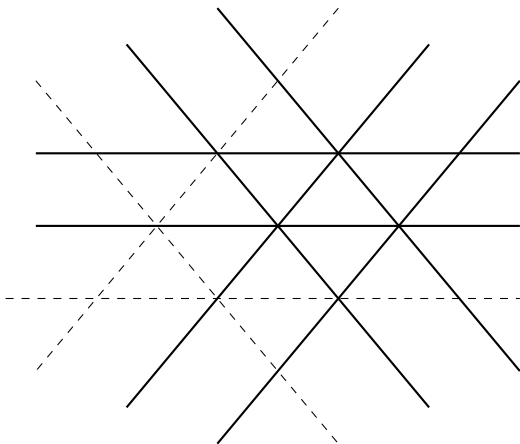
Bernardi's idea

Find m such that $\mathcal{A} \subseteq \mathcal{C}_n^{(m)}$. Choose a tree to represent each region of \mathcal{A} .



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- Catalan arrangement

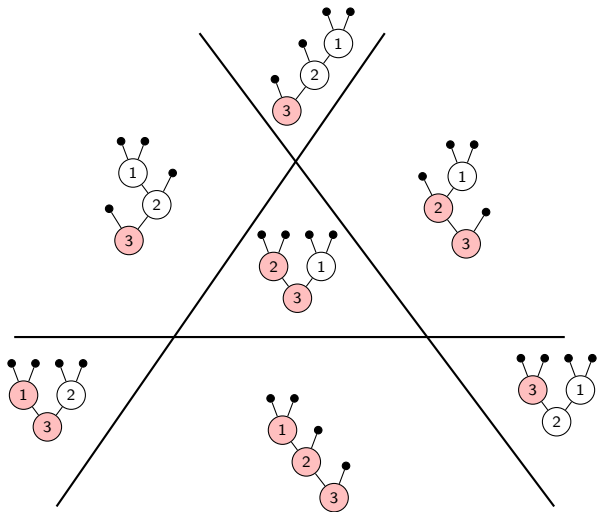
$$\mathcal{C}_n = \{x_i = x_j, x_i = x_j + 1 \mid i, j \in [n], i \neq j\}.$$

- Shi arrangement

$$\mathcal{S}_n = \{x_i = x_j, x_i = x_j + 1 \mid 1 \leq i < j \leq n\}.$$

- Linial arrangement

$$\mathcal{L}_n = \{x_i = x_j + 1 \mid 1 \leq i < j \leq n\}.$$



The characteristic polynomial of \mathcal{L}_3 is $t^3 - 3t^2 + 3t$.

Two-parameter Fuss-Catalan numbers

- Raney numbers

$$A_n(m, r) := \frac{r}{n(m+1) + r} \binom{n(m+1) + r}{n}$$

- Counts r -tuples of $(m+1)$ -ary trees with total number of nodes n .
- Setting $r = 1$ gives us Fuss-Catalan numbers.

Two-parameter Fuss-Catalan numbers

Question

Is there a family of arrangements with region numbers $n! \times A_n(m, r)$? If so, what is a corresponding statistic on r -tuples of trees?

- For $r = 1$, $\mathcal{C}_n^{(m)}$ and branch statistic.

- (Deshpande, M., Sarkar) For $r = 2$,

$$\mathcal{A}_n^{(m)} := \{x_i = 0 \mid i \in [n]\} \cup \{x_i = 2^k x_j \mid k \in [-m, m], 1 \leq i < j \leq n\}$$

and branch statistic on first tree of 2-tuple.

- For $r > 2$?

Thank You!