Decorated maps and quadrant tandem walks

Éric Fusy, Erkan Narmanli, and Gilles Schaeffer
Decorated maps

Graphs
Decorated maps

*Graphs*

*Maps*

*embedding on the plane*
Decorated maps

Graphs

Maps
embedding on the plane

≠
Decorated maps

Graphs

Maps
embedding on the plane
Decorated maps

Graphs

Maps
embedding on the plane
Decorated maps

Graphs

Maps
embedding on the plane
Decorated maps

→ exact enumeration formulas
→ universal critical exponent

# maps with $n$ edges : $\nu \cdot n^{-5/2}$

Maps embedding on the plane
Decorated maps

- exact enumeration formulas
- universal critical exponent
  
  \[ \# \text{ maps with } n \text{ edges} = \kappa \cdot \gamma^n n^{-5/2} \]

Maps
embedding on the plane

Decorated maps
orientation, coloration, etc.
Decorated maps

- exact enumeration formulas
- universal critical exponent
  \[ \# \text{ maps with } n \text{ edges} : \kappa \cdot \gamma^n n^{-5/2} \]

Maps embedding on the plane

- Schnyder woods
- transversal structures
- plane bipolar posets

Decorated maps orientation, coloration, etc.
Transversal structures
Transversal structures
Transversal structures
Link with rectangular tilings
Link with rectangular tilings
Link with rectangular tilings
Link with rectangular tilings
Link with rectangular tilings
Link with rectangular tilings

Regular edge labellings of 4-connected plane graphs and its applications in graph drawing problems, G. Kant & X. He (1997)
The KMSW Bijection

1. Application to three maps model
   a. Plane bipolar posets
   b. Plane bipolar posets by vertices
   c. Transversal structures
   d. Asymptotics

2. Interlude : plane permutations

3. Application to corner polyhedra
   a. Via polyhedral orientations
   b. Via Schnyder colorings
   c. Asymptotics
Plane bipolar orientation

Acyclic
1 single source S
1 single sink N
Plane bipolar orientation
Plane bipolar orientation

$\text{face of size } (i+1,j+1)$
The KMSW bijection

Plane bipolar orientations $\leftrightarrow$ tandems walks in the quarter plane
The KMSW bijection

Plane bipolar orientations \( \leftrightarrow \) tandems walks in the quarter plane
The KMSW bijection

Plane bipolar orientations \( \leftrightarrow \) tandems walks in the quarter plane

\[ a+1 \quad n+1 \quad b+1 \]

\[ \text{edges} \]

\[ S \quad N \]

\[ i, j > 0 \]

\[ a \quad b \quad n \]

\[ \text{Arrows} \]
The KMSW bijection

Plane bipolar orientations $\leftrightarrow$ tandems walks in the quarter plane

Diagram showing the bijection between plane bipolar orientations and tandems walks in the quarter plane.
The KMSW bijection

Plane bipolar orientations $\leftrightarrow$ tandems walks in the quarter plane
The KMSW bijection

Plane bipolar orientations ↔ tandems walks in the quarter plane

Start:
The KMSW bijection

Plane bipolar orientations \( \leftrightarrow \) tandems walks in the quarter plane

start :

end :
The KMSW bijection

**Plane bipolar orientations** ↔ **tandems walks in the quarter plane**
KMSW bijection example
KMSW bijection example
KMSW bijection example
KMSW bijection example
KMSW bijection example
KMSW bijection example
KMSW bijection example
KMSW bijection example
La bijection KMSW

\textit{bipolar orientations} $\leftrightarrow$ \textit{tandem walks in the quarter plane}
La bijection KMSW

Bipolar orientations \iff tandem walks in the quarter plane

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   b. Plane bipolar posets by vertices
   c. Transversal structures
   d. Asym tototics

2. Interlude: plane permutations

3. Application to corner polyhedra
   a. Via polyhedral orientations
   b. Via Schnyder colorings
   c. Asym tototics
Plane bipolar poset
Plane bipolar poset

Poset
(plane bipolar poset)

= Bipolar orientation
No multiple edge
Plane bipolar poset

Poset (plane bipolar poset)

= Bipolar orientation
No multiple edge
No transitive edge
Plane bipolar poset

Poset
(plane bipolar poset)

= Bipolar orientation
No multiple edge
No transitive edge
Plane bipolar poset

Poset (plane bipolar poset)

= Bipolar orientation
No multiple edge
No transitive edge

= Bipolar orientation
Specialization to Posets
Bipolar orientation where $i, j > 0$
Specialization to Posets

Bipolar orientation = \((i, j > 0)\)

Poset = }
The KMSW Bijection

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Specialization to Posets by vertices

Bipolar orientation
Specialization to Posets by vertices

Bipolar orientation
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Bipolar orientation

poset
Specialization to Posets by vertices
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poset
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poset
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1. Application to three maps model
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Specialization to transversal structures
Specialization to transversal structures
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Specialization to transversal structures
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# Specializations summary

<table>
<thead>
<tr>
<th>Model</th>
<th>Tandem walk</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Posets</strong>&lt;br&gt;n+2 edges</td>
<td><img src="image1.png" alt="Image" /></td>
</tr>
<tr>
<td><strong>Transversal structures</strong>&lt;br&gt;n blue edges</td>
<td><img src="image3.png" alt="Image" /></td>
</tr>
<tr>
<td><strong>Posets</strong>&lt;br&gt;n vertices</td>
<td><img src="image5.png" alt="Image" /></td>
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3. Application to corner polyhedra
   a. Via polyhedral orientations
   b. Via Schnyder colorings
   c. Asymtototics
Asymptotic counting results


Non-D-finite excursions in the quarter plane, D. Bostan, K. Raschel, B. Salvy (2012)

\[ a_n \sim \pi \cdot \gamma^n n^{-1 - \pi/\theta} \]
Asymptotic counting results

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\[ a_n \sim \kappa \cdot \gamma^n n^{-1 - \pi/\theta} \]

If the drift is zero, i.e.:

\[ \mathbb{E}[X] = \mathbb{E}[Y] = 0 \]

And the covariance matrix is identity.
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Asymptotic counting results

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<th>Asymptotics</th>
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</table>
| Posets $n+2$ edges | $e_n \sim \pi \gamma^n n^{-\alpha}$  
\[\alpha = -1 - \frac{\pi}{\arccos(\xi)}\]  
\[\gamma \approx 4.80 \ldots \quad \alpha \approx 5.14 \ldots\]  
$\gamma$ and $\xi$ are algebraic |
### Asymptotic counting results

<table>
<thead>
<tr>
<th>Model</th>
<th>Asymptotics</th>
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<tbody>
<tr>
<td><strong>Posets</strong> ( n+2 ) edges</td>
<td>( e_n \sim \kappa \gamma^n n^{-\alpha} ), ( \alpha = -1 - \frac{\pi}{\arccos(\xi)} )</td>
</tr>
<tr>
<td>( \gamma ) and ( \xi ) are algebraic ( \gamma \approx 4.80 \ldots ) ( \alpha \approx 5.14 \ldots )</td>
<td></td>
</tr>
<tr>
<td><strong>Transversal structures</strong> ( n ) vertices</td>
<td>( t_n \sim \kappa \left( \frac{27}{2} \right)^n n^{-1 - \frac{\pi}{\arccos(7/8)}} )</td>
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# Asymptotic counting results

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<tr>
<td></td>
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</table>
Asymptotic counting results

Model

$n+2$ edges and $n$ vertices

Posets

Transversal structures


$n$ vertices

Posets

$\frac{\pi}{\cos(\xi)}$

$\frac{n}{2}$

$\frac{n+\sqrt{5}}{2}$

$\sim \frac{1}{2}(n+\sqrt{5})^{n}n^{-6}$
Asymptotic counting results

**POSETS PER VERTEX**

\[ b_n \sim 2 \cdot \left( \frac{11 + \sqrt{5}}{2} \right)^n \cdot n^{-6} \]

**PLANE PERMUTATIONS**

\[ p_n \sim 2 \cdot \left( \frac{11 + \sqrt{5}}{2} \right)^n \cdot n^{-6} \]

Asymptotic counting results

Semi-Baxter and strong-Baxter: two relatives of Baxter Sequences,

Plane permutations

\[ p_n \sim 2 \left( \frac{11 + \sqrt{5}}{2} \right)^n \cdot n^{-6} \]

1, 1, 2, 6, 23, 104, 530, 2958, 1734, 112657, ...
Model Asymptotics

- n+2 edges and
- are algebraic

Posets

n vertices

Transversal structures


http://oeis.org/A117106

Asymptotic counting results


\[ p_n \approx \frac{2^n (n+5)^{n/2}}{\pi n} \]

Plane permutations

1, 1, 2, 6, 23, 104, 530, 2958, ...

1, 1, 2, 6, 23, 104, 530, 2958, ...

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1, 1, 2, 6, 23, 104, 530, 2958, 17734, 112657, 750726, ...

http://oeis.org/A117106

Todo: 

- Bijection
  - Poset per vertices
  - Plane permutations

Plane permutations

\[ p_n \sim \pi \left( \frac{11 + \sqrt{5}}{2} \right)^n \cdot n^{-6} \]
The KMSW Bijection

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Plane permutations
Plane permutations

Dominance relation
Plane permutations

Dominance relation
Plane permutations

Dominance diagram
= Dominance relation with no transitive edges
Plane permutations

Dominance diagram

= Dominance relation with no transitive edges
Plane permutation

$= \text{No edge crossing in the dominance diagram}$
Plane permutation

- No edge crossing in the dominance diagram

- Avoid the vincular pattern: 2 1 4 3
Link with plane permutations

Plane permutation $\rightarrow$ Poset
Link with plane permutations

Plane permutation $\rightarrow$ Poset

$\pi(i)$
Link with plane permutations

*Plane permutation* $\rightarrow$ *Poset*
Link with plane permutations

Plane permutation  →  Poset
Link with plane permutations

\[ \text{Poset} \quad \rightarrow \quad \text{Plane permutation} \]
Link with plane permutations

Poset $\rightarrow$ Plane permutation
Link with plane permutations

$\text{Poset} \longrightarrow \text{Plane permutation}$
Link with plane permutations

Poset $\rightarrow$ Plane permutation
Link with plane permutations

Poset $\rightarrow$ Plane permutation
Link with plane permutations

Poset \rightarrow Plane permutation
Link with plane permutations

Poset $\rightarrow$ Plane permutation
Link with plane permutations

Poset $\rightarrow$ Plane permutation
Link with plane permutations

Poset \rightarrow Plane permutation
Link with plane permutations

Poset  $\rightarrow$ Plane permutation
Link with plane permutations

Poset $\rightarrow$ Plane permutation
Link with plane permutations

Poset $\rightarrow$ Plane permutation
Link with plane permutations

Poset → Plane permutation

\[\pi: 1 \rightarrow 9\]
Link with plane permutations

$\text{Poset} \quad \longrightarrow \quad \text{Plane permutation}$

$\pi: \ 1 \rightarrow 9 \quad 2 \rightarrow 5$
Link with plane permutations

Poset $\longrightarrow$ Plane permutation

\[\pi: 1 \rightarrow 9, 2 \rightarrow 5, 3 \rightarrow 6\]
Link with plane permutations

Poset → Plane permutation

\[ \pi: \begin{align*}
1 & \rightarrow 9 \\
2 & \rightarrow 5 \\
3 & \rightarrow 6 \\
4 & \rightarrow 10 \\
5 & \rightarrow 7 \\
6 & \rightarrow 3 \\
7 & \rightarrow 4 \\
8 & \rightarrow 8 \\
9 & \rightarrow 1 \\
10 & \rightarrow 2
\end{align*} \]
Link with plane permutations

**Poset** → **Plane permutation**

Area requirement and symmetry display of planar upward drawings, G. Di Battista, R. Tamassia, and I. G. Tollis (1992)

\[ \pi: 1 \rightarrow 9 \quad 6 \rightarrow 3 \\
2 \rightarrow 5 \\
3 \rightarrow 6 \\
4 \rightarrow 10 \\
5 \rightarrow 7 \quad 10 \rightarrow 2 \]
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Corner polyhedra
Corner polyhedra
Corner polyhedra

Steinitz theorems for orthogonal polyhedra, D. Eppstein & E. Mumford (2010)
Corner polyhedra

Steinitz theorems for orthogonal polyhedra, D. Eppstein & E. Mumford (2010)

corner polyhedra ↔ polyhedral orientations
Corner polyhedra

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Steinitz theorems for orthogonal polyhedra,
D. Eppstein & E. Mumford (2010)
Polyhedral orientations
Polyhedral orientations
Polyhedral orientations
Polyhedral orientations
Polyhedral orientations
Polyhedral orientations
Polyhedral orientations

inner:

extremal $\times 2$
Polyhedral orientations

inner:

lateral

extremal x 2
Polyhedral orientations

inner:

lateral

extremal x 2
On Bipolar Orientations:
Polyhedral orientations

ON BIPOLAR ORIENTATIONS:
Polyhedral orientations

On bipolar orientations:

- Lateral
- Extremal

Diagram showing relationships between lateral and extremal orientations.
Polyhedral orientations

→ Steinitz theorems for orthogonal polyhera, D. Eppstein & E. Mumford (2010)
Polyhedral orientations

→ Steinitz theorems for orthogonal polyhera, D. Eppstein & E. Mumford (2010)
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Polyhedral orientations via KMSW
Polyhedral orientations via KMSW
Polyhedral orientations via KMSW
Polyhedral orientations via KMSW

when $x,y$

are even

when $i+j$

are even

$i \geq 2$

$j > 1$
Polyhedral orientations via KMSW

- When \(x, y\) are even
- When \(x, y\) are odd

\[
\begin{align*}
\text{when } x, y \text{ are even} & : i + j \text{ even} \\
\text{when } x, y \text{ are odd} & : i > 1, j > 1
\end{align*}
\]
## Bijections summary

<table>
<thead>
<tr>
<th>Model</th>
<th>Combinatorial type</th>
<th>Bipolar orient. / Tandem walk</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$ flats</td>
<td><img src="image" alt="Diagram" /></td>
<td></td>
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[Diagram of bipolar orientation and tandem walk]
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<td>n inner vertices</td>
<td>n + 1 edges</td>
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</table>

**Model**: 
- Representation of an arrangement of flats.

**Combinatorial Type**: 
- n inner vertices

**Bipolar Orient. / Tandem Walk**: 
- Diagram illustrating the relationship with arrows indicating direction and count.
### Bijectons summary

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Rigid corner polyhedra
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Schnyder labellings

Geodesic embeddings and planar graphs, S. Felsner (2003)
Schnyder labellings
Schnyder labellings
Schnyder labellings
Schnyder labellings

\[ \text{length 6} \]
Schnyder labellings
Schnyder labellings

Geodesic embeddings and planar graphs, S. Felsner (2003)

rigid corner polyhedra \rightleftharpoons \text{Schnyder labellings}
Schnyder labellings
Schnyder labellings
Schnyder labellings

\[ \text{All Schnyder Labellings} = \text{Hexagon} + \text{Hexagon} + \text{Hexagon} + \text{Hexagon} \]
Schnyder labellings

\[ \text{ALL SCHNYDER LABELLINGS} = \text{EXCEPT:} \]

\[ \text{except:} \]

\[ \text{except:} \]
Schnyder labellings
Schnyder labellings
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Schnyder labellings
Schnyder labellings

---

[Diagram of Schnyder labellings with colored nodes and lines]
Schnyder labellings
Schnyder labellings
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Schnyder labellings

**IT IS A BIPOLAR ORIENTATION:**

1) **ACYCLIC**
Schnyder labellings

1) ACYCLIC
ASSUME 3 CYCLE, TAKEN MINIMAL

IT IS A BIPOLAR ORIENTATION:
Schnyder labellings

IT IS A BIPOLAR ORIENTATION:

1) A CYCLIC
   ASSUME 3 CYCLE, TAKEN MINIMAL
1) **ACYCLIC**

Assume a cycle, taken minimal

**It is a bipolar orientation:**
1) ACYCLIC
ASSUME 3 CYCLE, TAKEN MINIMAL

IT IS A BIPOLAR ORIENTATION:
Schnyder labellings

IT IS A BIPOLAR ORIENTATION:

1) ACYCLIC
   ASSUME 3-CYCLE, TAKEN MINIMAL

REMINDER

[Diagram showing a cycle with arrows indicating the orientation]
Schnyder labellings

It is a bipolar orientation:

1) Acyclic
   Assume J cycle, taken minimal
It is a bipolar orientation:

1) Acyclic
Assume a cycle, taken minimal
Schnyder labellings

**IT IS A BIPOLAR ORIENTATION:**

1) **ACYCLIC**
   - Assume a cycle, taken minimal

**NOT MINIMAL**

REMINDER
Schnyder labellings

**It is a Bipolar Orientation:**

1) Acyclic
2) Single Sink/Source
Schnyder labellings

**IT IS A BIPOLAR ORIENTATION:**

1) **ACYCLIC**
2) **SINGLE SINK/SOURCE**
Schnyder labellings
Schnyder labellings
Schnyder labellings
Schnyder labellings
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Schnyder labellings
Schnyder labellings
Schnyder labellings
Schnyder labellings KMSW
Schnyder labellings KMSW

- When $x,y$ are even
- When $x,y$ are odd

- Diagram of a bipartite graph with arrows indicating labelings.
- Graphs showing conditions for $x,y$ being even or odd.

- $i+j$ even

- $j > 1, i > 2$ when $x,y$ are even

- $j > 2, i > 1$ when $x,y$ are odd
when $x,y$ are even

when $x,y$ are odd

$i = 2k+2$

$j = 2l+2$

regarding $x,y$ parity

Schnyder labellings KMSW
Schnyder labellings KMSW

when $x, y$ are even

when $x, y$ are odd

$i = 2k + 2$

$j = 2l + 2$

regarding $x, y$ parity

with

$i = 2k + 2$

$j = 2l + 2$

when $x, y$ are even

with

$i = 2k + 3$

$j = 2l + 1$

when $x, y$ are odd

with

$i = 2k + 1$

$j = 2l + 3$
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<td><img src="image6" alt="Bipolar Orient. / Tandem Walk" /></td>
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</table>
### Bijection Summary

<table>
<thead>
<tr>
<th>Model</th>
<th>Combinatorial type</th>
<th>Bipolar orient./Tandem walk</th>
</tr>
</thead>
<tbody>
<tr>
<td>n flats</td>
<td>n inner vertices</td>
<td>n+1 edges</td>
</tr>
<tr>
<td>n flats</td>
<td>n inner faces</td>
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Bijections summary

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<tr>
<td>$n$ flats</td>
<td>$n$ inner vertices</td>
<td>$n+1$ edges</td>
</tr>
<tr>
<td>$n$ flats</td>
<td>$n$ inner faces</td>
<td>$n+4$ vertices</td>
</tr>
</tbody>
</table>

Diagram showing bijections involving $n$ flats with $n$ inner vertices and $n+1$ edges, compared to $n$ flats with $n$ inner faces and $n+4$ vertices.
The KMSW Bijection

1. Application to three maps model
   a. Plane bipolar posets
   b. Plane bipolar posets by vertices
   c. Transversal structures
   d. Asymtototics

2. Interlude: plane permutations

3. Application to corner polyhedra
   a. Via polyhedral orientations
   b. Via Schnyder colorings
   c. Asymtototics
Asymptotic counting results
Asymptotic counting results
Asymptotic counting results
Asymptotic counting results

\[ \# \triangle \leq \# \left( \frac{9}{2} \right)^n \]

\[ \# \square \leq \left( \frac{16}{3} \right)^n \]
Asymptotic counting results

\[ \# \text{ of } n \leq \left(\frac{9}{2}\right)^n \]

\[ \# \text{ of } n \leq \left(\frac{16}{3}\right)^n \]
Asymptotic counting results

\[ \# \quad (9/2)^n \]

\[ \# \quad (16/3)^n \]
Asymptotic counting results

\[ \# \left\lfloor \frac{9}{2} \right\rfloor \leq \left( \frac{9}{2} \right)^n \]

\[ \# \left\lfloor \frac{16}{3} \right\rfloor \leq \left( \frac{16}{3} \right)^n \]
Asymptotic counting results

\# \begin{align*}
& \leq \left( \frac{9}{2} \right)^n \\
& \leq \left( \frac{16}{3} \right)^n
\end{align*}
Asymptotic counting results

\[ \# \triangle \leq \left( \frac{9}{2} \right)^n \]

\[ \# \text{hexagon} \leq \left( \frac{16}{3} \right)^n \]
Asymptotic counting results

\[ \# \text{ of } \text{triangle} \leq (9/2)^n \]

\[ \# \text{ of } \text{hexagon} \leq \left( \frac{16}{3} \right)^n \]
Asymptotic counting results

\[
\# \geq \# \quad \text{and} \quad n^{2/3} \leq \quad \text{and} \quad n^{2/3}
\]

\[
\# \leq \left(\frac{9}{2}\right)^n
\]

\[
\# \leq \left(\frac{16}{3}\right)^n
\]
Asymptotic counting results

\[
\left(9^{1/2}\right)^{n + o(n)} \leq \# \quad n \leq \left(9^{1/2}\right)^n
\]

\[
\left(16^{1/3}\right)^{n + o(n)} \leq \# \quad n \leq \left(16^{1/3}\right)^n
\]
Asymptotic counting results

\[ \left( \frac{9}{2} \right)^{n+o(n)} \leq \# \leq \left( \frac{9}{2} \right)^n \]

\[ \left( \frac{16}{3} \right)^{n+o(n)} \leq \# \leq \left( \frac{16}{3} \right)^n \]

\[ \lim_{n \to \infty} \frac{1}{p_n} = \frac{9}{2} \]

\[ \lim_{n \to \infty} \frac{1}{s_n} = \frac{16}{3} \]
## Summary

<table>
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<tr>
<th>Model</th>
<th>Tandem Walks</th>
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</table>
| **Transversal structures**  
  $n$ blue edges                            | ![Diagram](image1)  
  ![Diagram](image2)  
  $\binom{i+j-2}{i-1}$                     |
| **Polyhedral orientations**  
  $n$ inner faces                           | ![Diagram](image3)  
  ![Diagram](image4)  
  when $x,y$ are even                       |
| **Schnyder colorings**  
  $n$ inner faces                           | ![Diagram](image5)  
  ![Diagram](image6)  
  when $x,y$ are even                       |
| **Posets**  
  $n$ vertices                              | ![Diagram](image7)  
  ![Diagram](image8)  
  $\binom{k+1}{2}$                          |
| **Posets**  
  $n+2$ edges                               | ![Diagram](image9)  
  ![Diagram](image10)  
  $\binom{i+j}{i-1}$                        |
## Summary

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<th>Asymptotics</th>
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<tr>
<th>Model</th>
<th>Asymptotics</th>
<th>(a_n \sim \kappa \cdot \gamma^n n^\alpha)</th>
<th>(\alpha = 1 + \pi/\arccos(\xi))</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Transversal structures</strong></td>
<td></td>
<td>(\gamma = \frac{27}{2})</td>
<td>non D-Finite</td>
</tr>
<tr>
<td>(n) blue edges</td>
<td></td>
<td>(\alpha \approx 7.21)</td>
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<td>(\gamma = \frac{11 + 5\sqrt{5}}{2})</td>
<td>D-Finite</td>
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<tr>
<td>(n) vertices</td>
<td></td>
<td>(\alpha = 6)</td>
<td>(\xi = \frac{1 + \sqrt{5}}{4})</td>
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<tr>
<td><strong>Posets</strong></td>
<td></td>
<td>(\gamma \approx 4.80)</td>
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<td></td>
<td>(\alpha \approx 5.14)</td>
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<tr>
<td>$a + 1$</td>
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Merci