

Analytic solutions for walks with small steps in the three quarter plane

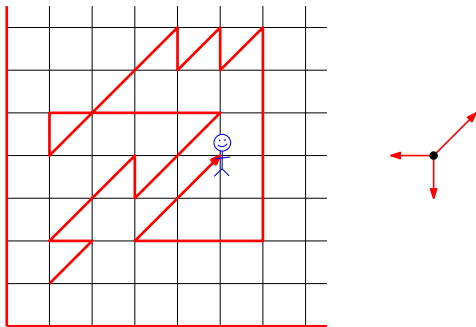
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INTRODUCTION: WALKS IN THE QUARTER PLANE

The model: Choose a step set $S \in \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$.



Problem: Determine the number $q_{n,i,j}$ of paths in the positive quadrant with steps in S from $(1, 1)$ to (i, j) of length n .

Equivalently: Determine the generating function

$$Q(t, x, y) := \sum_{n \geq 0} \sum_{i, j \geq 1} q_{n,i,j} t^n x^i y^j.$$

WALKS IN THE QUARTER PLANE BACKGROUND

Before 2010: Certain specific models considered and solved

[Kreweras, Gouyou-Beauchamps, Gessel]

Since then: 79 distinct non-trivial step sets identified

[Bousquet-Mélou, Mishna, 2010], systematic methods used to solve many cases

- Algebraic methods [Malyshev, Bousquet-Mélou, Mishna]
- Asymptotic analyses [Denisov, Wachtel, Mishna, Rechnitzer]
- Computer algebra [Bostan, Chyzak, Van Hoeij, Kauers, Pech]
- Galois Theory [Dreyfus, Hardouin, Roques, Singer]
- Analytic approach [Fayolle, Raschel, Kurkova, Bernardi]

All cases now characterised into complexity heirarchy

COMPLEXITY HEIRARCHY

For a series (or a function) $F(t)$, the following properties satisfy

Rational \Rightarrow Algebraic \Rightarrow D-finite \Rightarrow D-Algebraic :

- **Rational:** $F(t) = \frac{P(t)}{Q(t)}$ for polynomials $P(t)$ and $Q(t)$.
- **Algebraic:** $P(F(t)) = 0$ for some non-zero polynomial $P(x)$.
- **D-finite:** $F(t)$ satisfies some non-trivial linear differential equation. E.g.

$$t^3 F''(t) + t^2 F'(t) + (t + 1)F(t) - 1 = 0$$

- **D-algebraic:** $F(t)$ satisfies some non-trivial algebraic differential equation. E.g.

$$t^2 F'(t) + F'(t)F(t) + tF(t) = 0$$

For multivariate functions/series: Must be in the class (Rational, Algebraic, etc.) as a function of each variable.

QUADRANT WALKS

In total: 79 different non-trivial step sets S .

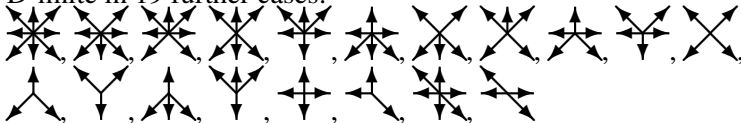
Complexity of generating function known precisely in each case:

$Q(t, x, y)$ is

- Algebraic in 4 cases:



- D-finite in 19 further cases:



- D-algebraic in 9 further cases:



- In remaining 47 cases, $Q(t, x, y)$ is not D-algebraic.

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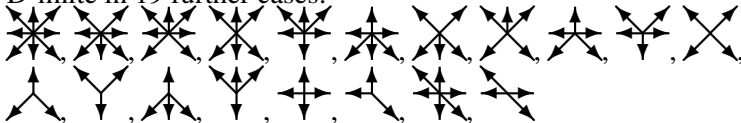
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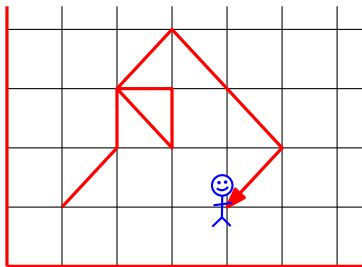
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This work: Same nature (in x) for 3/4-plane walks.

TALK OUTLINE

- **Part 1:** Walks in $1/4$ -plane
 - **Part 1a:** Deriving functional equation
 - **Part 1b:** Conversion to analytic functional equation
- **Part 2:** Walks in $3/4$ -plane
 - **Part 2a:** Deriving functional equation
 - **Part 2b:** Conversion to analytic functional equation
 - **Part 2c:** solution for walks starting on x -axis

Part 1a: Walks in 1/4-plane functional equation



QUADRANT WALKS WITH SMALL STEPS

Concept: The walker dies when they touch an axis.

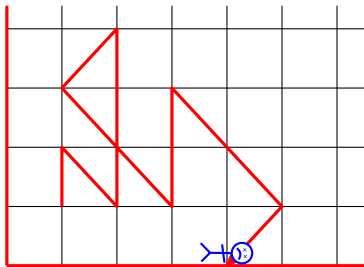
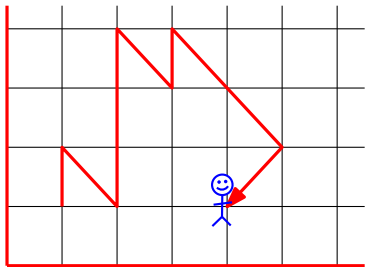
Quadrant walk: Walk from $(1, 1)$ not touching an axis.

Dead walk: Walk from $(1, 1)$ only touching an axis at the end.

(**Quadrant walk**+step) or **empty walk** = **Quadrant walk** or **Dead walk**

Generating function $D(x, y; t)$ for dead walks splits as

$$D(x, y; t) = A(x; t) + B(y; t) + C(t).$$



Unknowns: $Q(x, y; t)$, $A(x; t)$, $B(y; t)$, $C(t)$.

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Define the single-step polynomial: $P(x, y) = \sum_{(i,j) \in S} x^i y^j$.

$$Q(x, y; t)tP(x, y) + xy = Q(x, y; t) + D(x, y; t).$$

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To simplify, write Kernel $K(x, y; t) = 1 - tP(x, y)$.

To solve:

$$xy = K(x, y; t)Q(x, y; t) + A(x; t) + B(y; t) + C(t).$$

Unknowns: $Q(x, y; t), A(x; t), B(y; t), C(t)$.

Part 1b: Algebraic functional equation \rightarrow analytic functional equation

Originally from [Fayolle,Ianogorodski,Malyshev,1999] and [Raschel, 2010]
Used in [Fayolle, Raschel, 2010], [Kurkova, Raschel, 2012], [Bernardi, Bousquet-Mélou, Raschel, 2021], [Dreyfus,Hardouin,Roques,Singer, 2020], etc.

QUADRANT WALKS SOLUTION

To solve: (for A , B , C and hence Q)

$$xy = K(x, y; t)Q(x, y; t) + A(x; t) + B(y; t) + C(t).$$

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$$xy = K(x, y; t)Q(x, y; t) + A(x; t) + B(y; t) + C(t).$$

Solution idea:

Step 1: Fix $t < 1/9$ and define $\mathcal{K} = \{(x, y) : K(x, y; t) = 0\}$. Then for $(x, y) \in \mathcal{K}$, and $|x|, |y| < 1$, we have

$$A(x) + B(y) + C = xy.$$

Classic theorem: \mathcal{K} is a surface with genus 1 (usually), so is homeomorphic to some $\mathbb{C}/(\pi\mathbb{Z} + \pi\tau\mathbb{Z})$, where $\tau \in i\mathbb{R}$.

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Step 2: Parametrise \mathcal{K} using functions $X, Y : \mathbb{C} \rightarrow \mathbb{C} \cup \infty$, satisfying for some $\gamma \in i\mathbb{R}$:

- $X(z) = X(z + \pi) = X(z + \pi\tau) = X(-\gamma - z)$
- $Y(z) = Y(z + \pi) = Y(z + \pi\tau) = Y(\gamma - z)$

To solve: (for A, B and C)

$$A(X(z)) + B(Y(z)) + C = X(z)Y(z) \quad \text{when } |X(z)|, |Y(z)| < 1.$$

MORE ABOUT $X(z)$ AND $Y(z)$

$X(z)$ and $Y(z)$ are determined directly by the step set S .

By the definitions, X and Y satisfy

- $\sum_{(i,j) \in S} X(z)^i Y(z)^j = t^{-1}$ (that is, $K(X(z), Y(z)) = 0$),
- $X(z) = X(z + \pi) = X(z + \pi\tau) = X(-\gamma - z)$,
- $Y(z) = Y(z + \pi) = Y(z + \pi\tau) = Y(\gamma - z)$.

They can be written as

$$\begin{aligned} X(z) &= x_c \frac{\vartheta(z - \alpha, \tau) \vartheta(z + \gamma + \alpha, \tau)}{\vartheta(z - \delta, \tau) \vartheta(z + \gamma + \delta, \tau)} \\ &= x_c \prod_{k=-\infty}^{\infty} \frac{\sin(z - \alpha + k\pi\tau) \sin(z + \gamma + \alpha + k\pi\tau)}{\sin(z - \delta + k\pi\tau) \sin(z + \gamma + \delta + k\pi\tau)}, \\ Y(z) &= y_c \frac{\vartheta(z - \beta, \tau) \vartheta(z + \gamma + \beta, \tau)}{\vartheta(z - \epsilon, \tau) \vartheta(z + \gamma + \epsilon, \tau)} \\ &= y_c \prod_{k=-\infty}^{\infty} \frac{\sin(z - \beta + k\pi\tau) \sin(z - \gamma + \beta + k\pi\tau)}{\sin(z - \epsilon + k\pi\tau) \sin(z - \gamma + \epsilon + k\pi\tau)}. \end{aligned}$$

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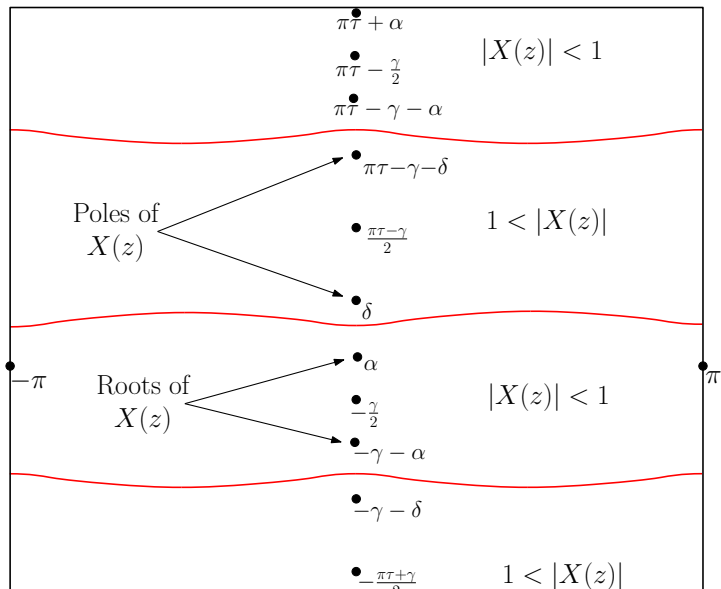
$$X(z) = x_0 \frac{\wp(z + \frac{\gamma}{2}, \pi, \pi\tau) + a}{\wp(z + \frac{\gamma}{2}, \pi, \pi\tau) + d},$$
$$Y(z) = y_0 \frac{\wp(z - \frac{\gamma}{2}, \pi, \pi\tau) + b}{\wp(z - \frac{\gamma}{2}, \pi, \pi\tau) + e}.$$

Note: there are 8 unknowns ($\tau, \gamma, x_0, y_0, a, b, d$ and e).

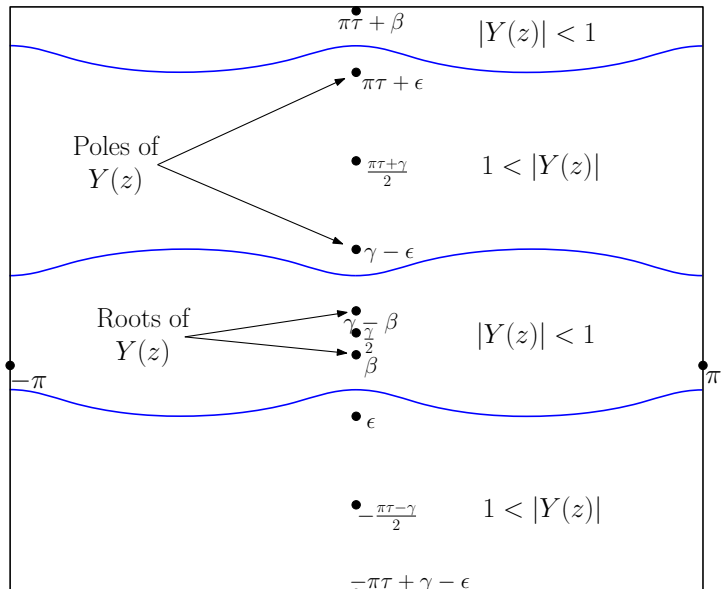
Corresponds to: Dimension of space \mathcal{L}/\mathbb{R} where \mathcal{L} is the space of

polynomials $\sum_{i,j \in \{-1,0,1\}} a_{i,j} x^i y^j$.

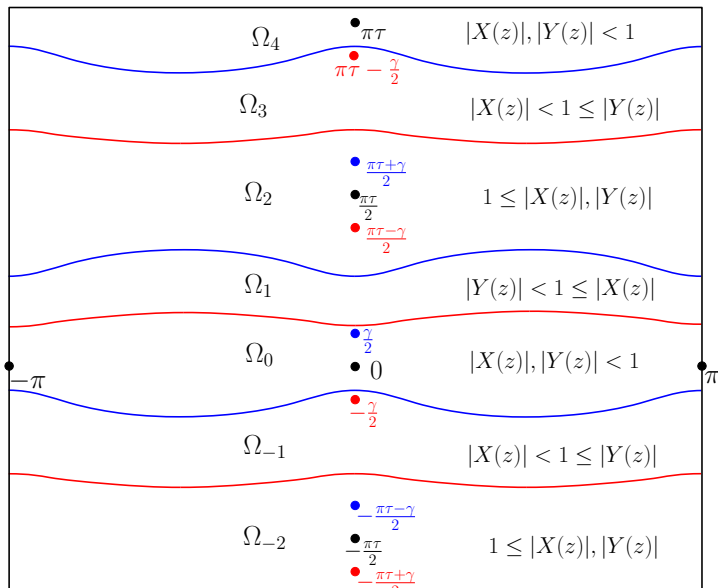
MORE ABOUT $X(z)$



MORE ABOUT $Y(z)$



MORE ABOUT $X(z)$ AND $Y(z)$



QUADRANT WALKS SOLUTION

Recall: We substituted $x \rightarrow X(z)$ and $y \rightarrow Y(z)$ into the equation

$$xy = Q(x, y)K(x, y) + A(x) + B(y) + C,$$

for quadrant walks. Only for $|X(z)|, |Y(z)| < 1$. e.g., for $z \in \Omega_0$.

To solve: (for A , B and C)

$$A(X(z)) + B(Y(z)) + C = X(z)Y(z) \quad \text{for } z \in \Omega_0.$$

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To solve: (for A , B and C)

$$A(X(z)) + B(Y(z)) + C = X(z)Y(z) \quad \text{for } z \in \Omega_0.$$

Step 3: Define

$$\tilde{A}(z) = A(X(z)), \quad \text{for } z \in \Omega_0 \cup \Omega_{-1},$$

$$\tilde{B}(z) = B(X(z)), \quad \text{for } z \in \Omega_0 \cup \Omega_1.$$

To solve:

$$\tilde{A}(z) + \tilde{B}(z) + C = X(z)Y(z),$$

given that

$$\tilde{A}(z) = \tilde{A}(z + \pi) = \tilde{A}(-\gamma - z),$$

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Other details needed:

- \tilde{A} and \tilde{B} are holomorphic
- \tilde{A} and \tilde{B} extend to meromorphic functions on \mathbb{C}
- \tilde{A} has roots at roots α and $-\gamma - \alpha$ of $X(z)$
- \tilde{B} has roots at roots β and $\gamma - \beta$ of $Y(z)$

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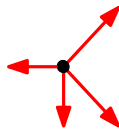
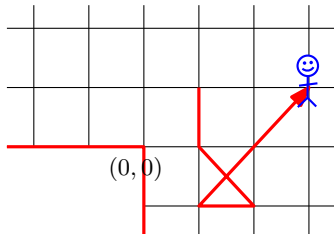
Using this method:

- Exact integral expression solution for $Q(x, y; t)$ [Raschel, 2010]
- Nature of $Q(x, y; t)$ as a function of x and y
- In D-algebraic cases: prove $Q(x, y; t)$ is D-algebraic in all variables [Bernardi, Bouquet-Mélou, Raschel, 2021]

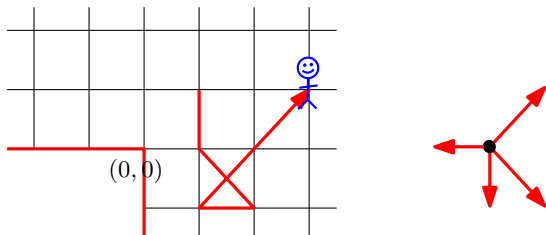
To do:

- Determine nature of $Q(x, y; t)$ as a function of t in all cases
- Determine asymptotic form of solution

Part 2: 3/4-plane



3/4-PLANE WALKS PREVIOUS RESEARCH

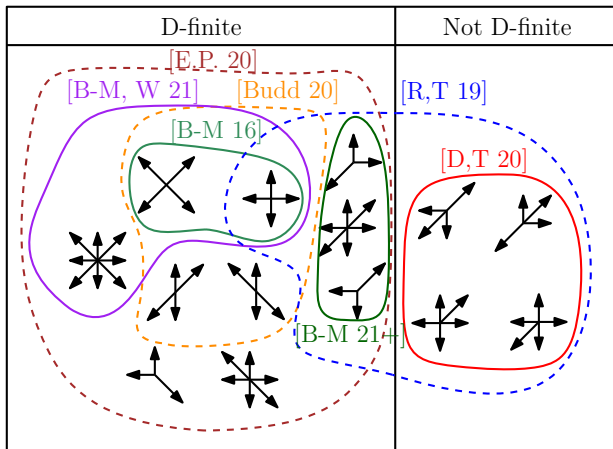


Death red line: Negative axes

Now only 74 distinct interesting cases.

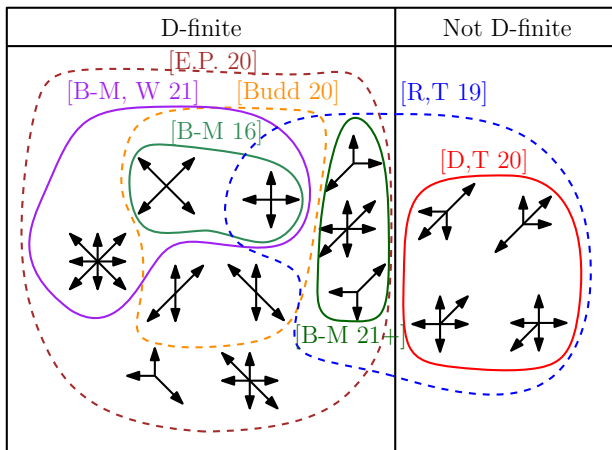
Nature (Algebraic/D-finite/D-algebraic) previously determined in 14 cases. Always the same as in quarter plane.

WALKS IN THE 3/4-PLANE: PREVIOUS SOLVED CASES



[Bousquet-Mélou 16], [Raschel, Trotignon 19], [Budd 20], [E.P., 20], [Dreyfus, Trotignon, 21], [Bousquet-Mélou, Wallner 21], [Bousquet-Mélou 21+].

WALKS IN THE 3/4-PLANE: PREVIOUS SOLVED CASES



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This work: remaining 60 cases still have the same nature (w.r.t. x)

PREVIEW: ANALYTIC FUNCTIONAL EQUATION

Equation to solve in 1/4-plane:

$$\tilde{A}(z) + \tilde{B}(z) = X(z)Y(z),$$

given that

$$\begin{aligned}\tilde{A}(z) &= \tilde{A}(z + \pi) = \tilde{A}(-\gamma - z), \\ \tilde{B}(z) &= \tilde{B}(z + \pi) = \tilde{B}(\gamma - z).\end{aligned}$$

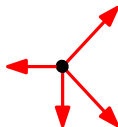
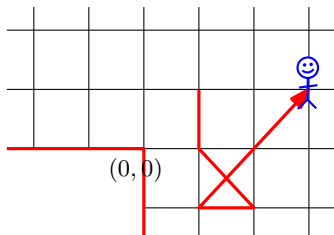
Equation to solve in 3/4-plane:

$$\hat{A}(z) + \hat{B}(z) = X(z)Y(z),$$

given that

$$\begin{aligned}\hat{A}(z) &= \hat{A}(z + \pi) = \hat{A}(\pi\tau - \gamma - z), \\ \hat{B}(z) &= \hat{B}(z + \pi) = \hat{B}(-\pi\tau + \gamma - z).\end{aligned}$$

Part 2a: Walks in 3/4-plane functional equation



THREE QUADRANT CONE WALKS WITH SMALL STEPS

Concept: The walker dies when they touch the red line.

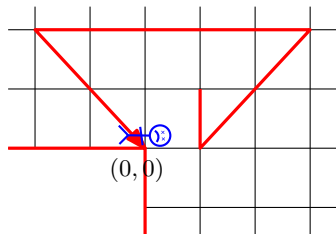
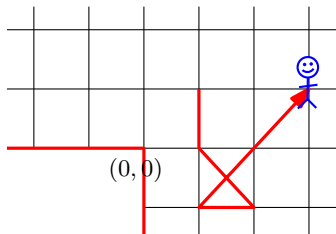
Cone walk: Walk from $(1, 1)$ not touching red line.

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(Cone walk + step) or empty walk = Cone walk or Dead walk

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Unknowns: $C(x, y; t), A(x; t), B(y; t), C(t)$.

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Unknowns: $C(x, y; t), A(x; t), B(y; t), C(t)$.

Part 2b: Algebraic functional
equation \rightarrow analytic functional
equation

3 QUADRANT CONE WALKS SOLUTION

To solve: (for A , B , C and hence \mathbf{C})

$$xy = K(x, y; t)C(x, y; t) + A(x; t) + B(y; t) + C(t).$$

3 QUADRANT CONE WALKS SOLUTION

To solve: (for A , B , C and hence C)

$$xy = K(x, y; t)C(x, y; t) + A(x; t) + B(y; t) + C(t).$$

Solution idea:

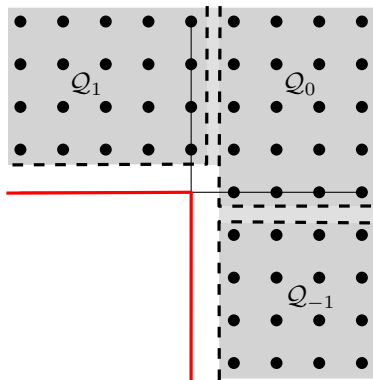
Step 1: Fix $t < 1/9$ and define $X(z)$ and $Y(z)$ as before (so $K(X(z), Y(z)) = 0$).

Step 2: Try to substitute $x \rightarrow X(z)$ and $y \rightarrow Y(z)$.

Step 3: Panic because $C(X(z), Y(z); t)$ never converges (it contains positive and negative powers of both $X(z)$ and $Y(z)$).

3 QUADRANT CONE WALKS SOLUTION FIX

Step 1: split the $3/4$ plane into quadrants \mathcal{Q}_{-1} , \mathcal{Q}_0 and \mathcal{Q}_1 .



Definition: Generating functions $\mathcal{Q}_{-1}(x, y^{-1}; t)$, $\mathcal{Q}_0(x, y; t)$, $\mathcal{Q}_1(x^{-1}, y; t)$ count walks ending in \mathcal{Q}_{-1} , \mathcal{Q}_0 , \mathcal{Q}_1 respectively.

$$\mathbf{C}(x, y; t) = \mathcal{Q}_{-1}(x, y^{-1}; t) + \mathcal{Q}_0(x, y; t) + \mathcal{Q}_1(x^{-1}, y; t).$$

3 QUADRANT CONE WALKS SOLUTION

Old equation to solve: (for A , B , C and hence Q)

$$xy = K(x, y)C(x, y) + A(x^{-1}) + B(y^{-1}) + C.$$

splits into **new equations to solve:**

$$-K(x, y)Q_{-1}\left(x, \frac{1}{y}\right) = B\left(\frac{1}{y}\right) + H_1(x) + \frac{1}{y}H_2(x)$$

$$-K(x, y)Q_0(x, y) = -xy + C - V_1(y) - xV_2(y) - H_1(x) - \frac{1}{y}H_2(x)$$

$$-K(x, y)Q_1\left(\frac{1}{x}, y\right) = A\left(\frac{1}{x}\right) + V_1(y) + xV_2(y).$$

3 QUADRANT CONE WALKS SOLUTION

Old equation to solve: (for A , B , C and hence Q)

$$xy = K(x, y)C(x, y) + A(x^{-1}) + B(y^{-1}) + C.$$

splits into **new equations to solve:**

$$-K(x, y)Q_{-1}\left(x, \frac{1}{y}\right) = B\left(\frac{1}{y}\right) + H_1(x) + \frac{1}{y}H_2(x) \quad \text{for } |x| < 1 < |y|$$

$$-K(x, y)Q_0(x, y) = -xy + C - V_1(y) - xV_2(y) - H_1(x) - \frac{1}{y}H_2(x) \quad \text{for } |x|, |y| < 1$$

$$-K(x, y)Q_1\left(\frac{1}{x}, y\right) = A\left(\frac{1}{x}\right) + V_1(y) + xV_2(y). \quad \text{for } |y| < 1 < |x|$$

These do converge in useful regions.

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These do converge in useful regions.

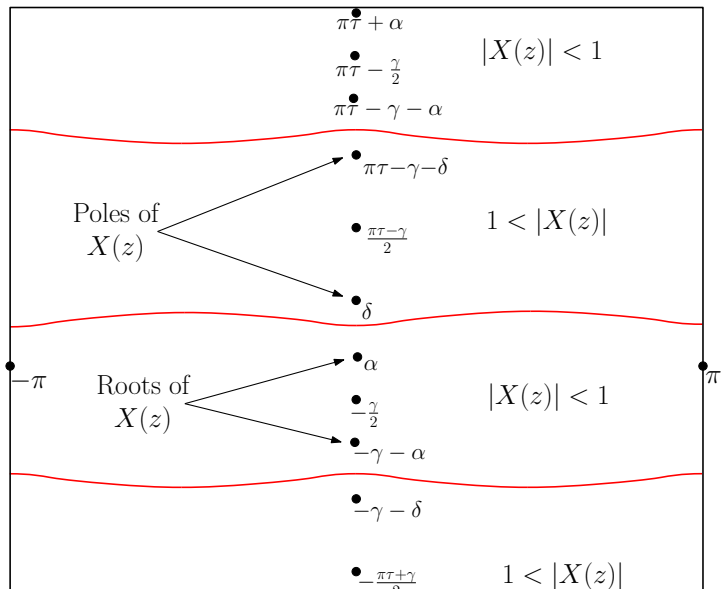
Next step: Substitute $x \rightarrow X(z)$ and $y \rightarrow Y(z)$ in each equation:

$$0 = B\left(\frac{1}{Y(z)}\right) + H_1(X(z)) + \frac{1}{Y(z)}H_2(X(z)) \quad \text{for } |X(z)| < 1 < |Y(z)|$$

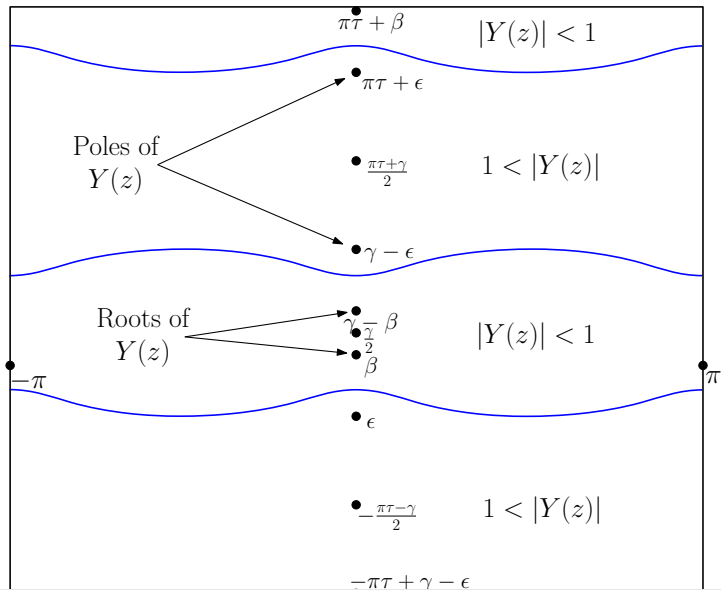
$$0 = -X(z)Y(z) + C - V_1(Y(z)) - X(z)V_2(Y(z)) - H_1(X(z)) - \frac{1}{Y(z)}H_2(X(z)) \quad \text{for } |X(z)|, |Y(z)| < 1$$

$$0 = A\left(\frac{1}{X(z)}\right) + V_1(Y(z)) + X(z)V_2(Y(z)). \quad \text{for } |Y(z)| < 1 < |X(z)|$$

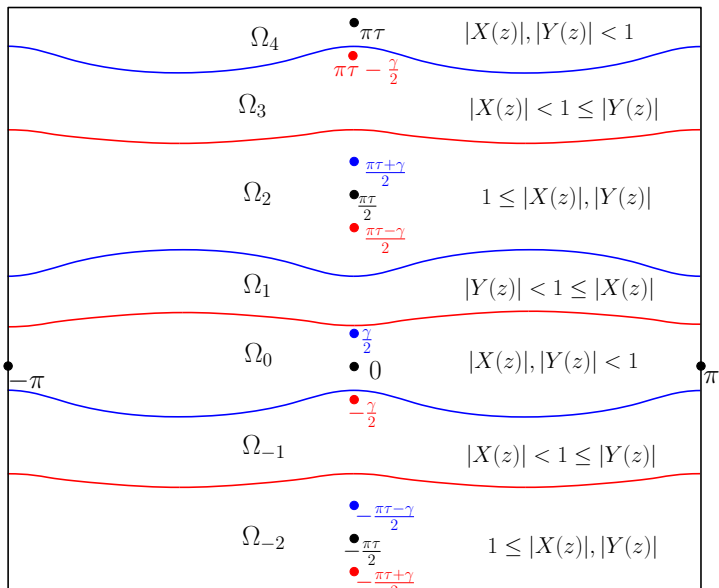
RECALL: MORE ABOUT $X(z)$



RECALL: MORE ABOUT $Y(z)$



RECALL: MORE ABOUT $X(z)$ AND $Y(z)$



3 QUADRANT CONE WALKS SOLUTION

Equations to solve:

$$-K(x, y)Q_{-1}\left(x, \frac{1}{y}\right) = B\left(\frac{1}{y}\right) + H_1(x) + \frac{1}{y}H_2(x)$$

$$-K(x, y)Q_0(x, y) = -xy + C - V_1(y) - xV_2(y) - H_1(x) - \frac{1}{y}H_2(x)$$

$$-K(x, y)Q_1\left(\frac{1}{x}, y\right) = A\left(\frac{1}{x}\right) + V_1(y) + xV_2(y).$$

3 QUADRANT CONE WALKS SOLUTION

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$$-K(x, y)Q_{-1}\left(x, \frac{1}{y}\right) = B\left(\frac{1}{y}\right) + H_1(x) + \frac{1}{y}H_2(x) \quad \text{for } |x| < 1 < |y|$$

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$$-K(x, y)Q_1\left(\frac{1}{x}, y\right) = A\left(\frac{1}{x}\right) + V_1(y) + xV_2(y). \quad \text{for } |y| < 1 < |x|$$

Define:

$$\hat{B}(z) := B\left(\frac{1}{Y(z)}\right), \quad \text{for } z \in \Omega_{-2} \cup \Omega_{-1},$$

$$\hat{H}(z) := H_1(X(z)) + \frac{1}{Y(z)}H_2(X(z)), \quad \text{for } z \in \Omega_{-1} \cup \Omega_0,$$

$$\hat{V}(z) := V_1(Y(z)) + X(z)V_2(Y(z)), \quad \text{for } z \in \Omega_0 \cup \Omega_1,$$

$$\hat{A}(z) := A\left(\frac{1}{X(z)}\right), \quad \text{for } z \in \Omega_1 \cup \Omega_2.$$

Then

$$0 = \hat{B}(z) + \hat{H}(z), \quad \text{for } z \in \Omega_{-1},$$

$$0 = C - X(z)Y(z) - \hat{H}(z) - \hat{V}(z), \quad \text{for } z \in \Omega_0,$$

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3 QUADRANT CONE WALKS SOLUTION

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Then

$$0 = \hat{B}(z) + \hat{H}(z), \quad \text{for } z \in \Omega_{-1},$$

$$0 = C - X(z)Y(z) - \hat{H}(z) - \hat{V}(z), \quad \text{for } z \in \Omega_0,$$

$$0 = \hat{A}(z) + \hat{V}(z), \quad \text{for } z \in \Omega_1.$$

To solve: (Sum of equations on right)

$$0 = C - X(z)Y(z) + \hat{A}(z) + \hat{B}(z).$$

From definitions:

$$\hat{A}(z) = \hat{A}(z + \pi) = \hat{A}(\pi\tau - \gamma - z),$$

$$\hat{B}(z) = \hat{B}(z + \pi) = \hat{B}(-\pi\tau + \gamma - z).$$

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Other details needed:

- \hat{A} is holomorphic on $\Omega_0 \cup \Omega_1 \cup \Omega_2$
- \hat{B} is holomorphic on $\Omega_0 \cup \Omega_{-1} \cup \Omega_{-2}$
- \hat{A} and \hat{B} are meromorphic on \mathbb{C}
- \hat{A} has roots at poles δ and $\pi\tau - \gamma - \delta$ of $X(z)$
- \hat{B} has roots at poles ϵ and $-\pi\tau + \gamma - \epsilon$ of $Y(z)$

3 QUADRANT CONE WALKS SOLUTION

To solve:

$$0 = C - X(z)Y(z) + \hat{A}(z) + \hat{B}(z).$$

From definitions:

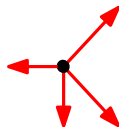
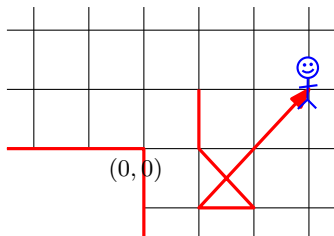
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$$\hat{B}(z) = \hat{B}(z + \pi) = \hat{B}(-\pi\tau + \gamma - z).$$

Conclusion: As in quarter plane case we get:

- Exact integral expression solution for $Q(x, y; t)$
- Nature of $Q(x, y; t)$ as a function of x and y
- In D-algebraic cases: proof $Q(x, y; t)$ is D-algebraic in all variables

Part 2c: Walks in 3/4-plane walks starting on x -axis



WALKS STARTING AT $(p, 0)$

To solve:

$$X(z)^p = C + \hat{A}_p(z) + \hat{B}_p(z).$$

where

$$\hat{A}_p(z) = \hat{A}_p(z + \pi) = \hat{A}_p(\pi\tau - \gamma - z),$$

$$\hat{B}_p(z) = \hat{B}_p(z + \pi) = \hat{B}_p(-\pi\tau + \gamma - z).$$

WALKS STARTING AT $(p, 0)$

To solve:

$$X(z)^p = C + \hat{A}_p(z) + \hat{B}_p(z).$$

where

$$\begin{aligned}\hat{A}_p(z) &= \hat{A}_p(z + \pi) = \hat{A}_p(\pi\tau - \gamma - z), \\ \hat{B}_p(z) &= \hat{B}_p(z + \pi) = \hat{B}_p(-\pi\tau + \gamma - z).\end{aligned}$$

and

- \hat{A}_p is holomorphic on $\Omega_0 \cup \Omega_1 \cup \Omega_2$
- \hat{B}_p is holomorphic on $\Omega_0 \cup \Omega_{-1} \cup \Omega_{-2}$
- \hat{A}_p and \hat{B}_p are meromorphic on \mathbb{C}
- \hat{A}_p has roots at poles δ and $\pi\tau - \gamma - \delta$ of $X(z)$
- \hat{B}_p has roots at poles ϵ and $-\pi\tau + \gamma - \epsilon$ of $Y(z)$

WALKS STARTING AT $(p, 0)$

To solve:

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where

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Rearranging equation:

$$\hat{B}_p(z) = X(z)^p - C - \hat{A}_p(z),$$

which is fixed under $z \rightarrow \pi\tau - \gamma - z$. So,

$$\hat{B}_p(z) = \hat{B}_p(z + \pi) = \hat{B}_p(2\pi\tau - 2\gamma + z)$$

WALKS STARTING AT $(p, 0)$

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Rearranging equation:

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$$\hat{B}_p(z) = \hat{B}_p(z + \pi) = \hat{B}_p(2\pi\tau - 2\gamma + z)$$

→ Exact solution for \hat{B}_p → exact solution for $C(x, y; t)$

Surprising result: Relation between walks starting at $(1, 0)$ and walks starting at $(p, 0)$.

WALKS STARTING AT $(1, 0)$ SOLUTION FOR $B_1(z)$

For $B_1(z)$:

$$\begin{aligned} B_1(z) &= b \frac{\vartheta(z - \epsilon, 2\tau - 2\frac{\gamma}{\pi})\vartheta(z + \pi\tau + \epsilon - \gamma, 2\tau - 2\frac{\gamma}{\pi})}{\vartheta(z - \delta, 2\tau - 2\frac{\gamma}{\pi})\vartheta(z + \pi\tau + \delta - \gamma, 2\tau - 2\frac{\gamma}{\pi})} \\ &= b \prod_{j=-\infty}^{\infty} \frac{\sin(z - \epsilon + 2k(\pi\tau - \gamma)) \sin(z + \epsilon + (2k + 1)(\pi\tau - \gamma))}{\sin(z - \delta + 2k(\pi\tau - \gamma)) \sin(z + \delta + (2k + 1)(\pi\tau - \gamma))} \end{aligned}$$

Then $B_p(z)$ is a degree p polynomial of $B_1(z)$

WALKS STARTING AT $(p, 0)$ SOLUTION.

Definition: $A_p\left(\frac{1}{x}; t\right)$ is the generating function for killed 3/4-plane walks from $(p, 0)$ to the non-positive x -axis.

Theorem part A: For each p , there is a degree p polynomial $H_p(a; t) \in a\mathbb{Z}[[t]][[a]]$ satisfying

$$A_p\left(\frac{1}{x}; t\right) = x^p - H_p\left(x - A_1\left(\frac{1}{x}; t\right); t\right).$$

H_p is uniquely determined by the fact that $A_p\left(\frac{1}{x}; t\right) \in \mathbb{Z}[[\frac{1}{x}, t]]$.

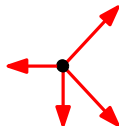
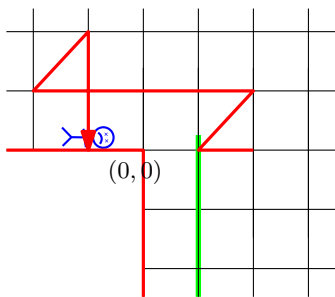
Theorem part B: the generating function $B_p\left(\frac{1}{y}; t\right)$ for killed walks from $(p, 0)$ to the negative y -axis is

$$B_p\left(\frac{1}{y}; t\right) = H_p\left(B_1\left(\frac{1}{y}; t\right); t\right).$$

→ surprising result even for $p = 2$

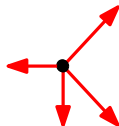
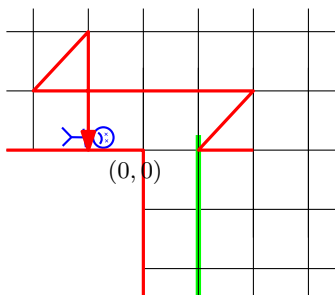
3/4 WALKS FROM $(2, 0)$ TO $(-1, 0)$.

Surprising result: For any step set S and any length n , Exactly half of the killed walks in the 3/4-plane from $(2, 0)$ to $(-1, 0)$ touch the green ray $\{(1, y) : y \leq 0\}$.



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Bijection??

