

Generalised Kontsevich graphs: topological recursion and r-spin intersection numbers

Séverin Charbonnier

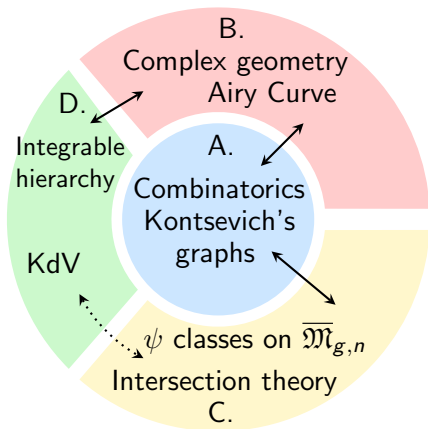
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Witten's conjecture ('90): the generating series of intersection numbers of ψ -classes on the moduli spaces of stable curves

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n}$$

is a tau-function of the KdV hierarchy (infinite set of non-linear differential equations).

Kontsevich ('92): generating functions of $\langle \tau_{d_1} \dots \tau_{d_n} \rangle_g =$ volumes of the moduli spaces of Riemann surfaces = sums over weighted fatgraphs (Kontsevich's graphs).

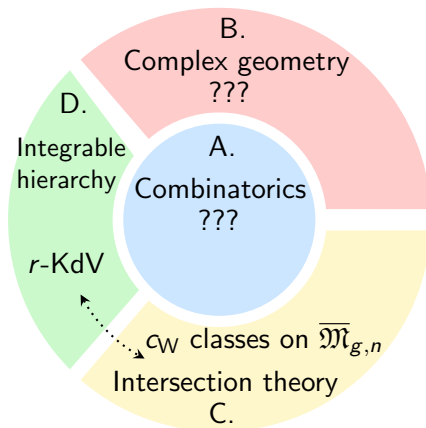
Kontsevich ('92): the enumeration of the fatgraphs is solved by (what was later called) topological recursion on the Airy curve.

Dijkgraaf–Verlinde–Verlinde ('91): topological recursion on the Airy curve \Leftrightarrow Virasoro constraints of the KdV hierarchy.

Theorem (Witten's conjecture/Kontsevich's theorem)

The generating series of intersection numbers of ψ -classes on the moduli spaces of stable curves is a tau-function of the KdV hierarchy.

Question: what is the complete picture if we consider Witten's conjecture about r -spin classes and the r -KdV hierarchy ('93)?



Combinatorial side? Topological recursion side?

Plan

- ① Combinatorial model: definitions and preliminary identities
 - Definition of the model
 - Origin of the model: matrix models
 - Combinatorial identities

- ② Computing the generating series: Tutte's equation and Topological Recursion
 - Tutte's recursion for GKG
 - Topological recursion

- ③ Application: r -spin intersection numbers satisfy Topological Recursion

Table of contents

- ① Combinatorial model: definitions and preliminary identities
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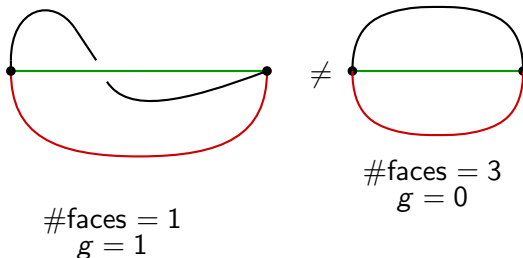
- ② Computing the generating series: Tutte's equation and Topological Recursion
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Definition (Map)

- A **map** is a graph G endowed with a cyclic order on the edges incident to each vertex.
- **Genus** g of a connected map:

$$\#\text{vertices} - \#\text{edges} + \#\text{faces} = 2 - 2g$$



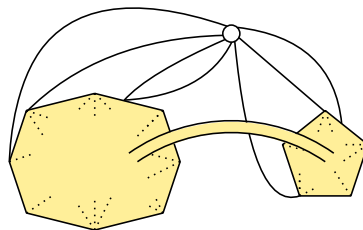
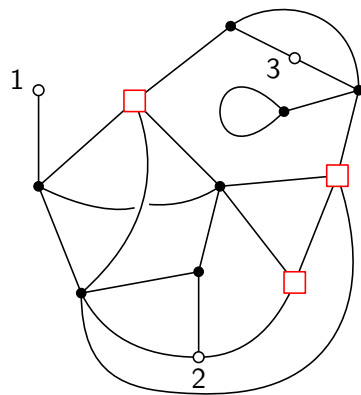
Generalised Kontsevich Graphs

Let $r \geq 2$.

Definition (GKG)

A **Generalised Kontsevich Graph** (GKG) is a connected map which can carry 3 types of vertices:

- Black vertices, of degree $3 \leq d \leq r + 1$;
- Labelled white vertices (from 1 to n), of degree $d \geq 1$, with the constraints:
 - (a) each face is adjacent to at most one white vertex;
 - (b) if a white vertex and its half-edges are removed, the remaining half-edges – that where attached to the erased set – belong to the same face (**star constraint**);
- Square vertices, of degree $d \geq 1$.



Decorated GKG

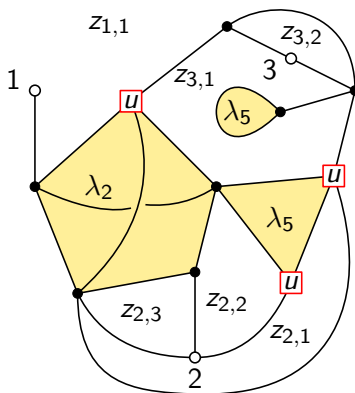
Faces adjacent to white vertices of a GKG are called **marked faces**, the other faces are **unmarked faces**.

Definition (Decorated Generalised Kontsevich Graphs)

Let $N \in \mathbb{N}$, $\lambda_1, \dots, \lambda_N \in \mathbb{C}$, and let G be a GKG with n white vertices.

G is a **decorated Generalised Kontsevich Graph** (decorated GKG) if:

- the marked faces around the j^{th} white vertex carry the complex variables $z_{j,1}, \dots, z_{j,k_j}$;
- each unmarked face carries a parameter belonging to $\{\lambda_1, \dots, \lambda_N\}$;
- each square vertex carries the complex variable u .



Sets of interest

We are interested in the weighted enumeration of graphs of the following set:

Definition (ciliated GKG)

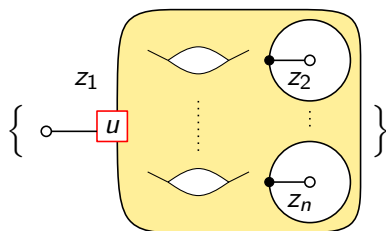
$G \in \mathcal{W}_{g,n}^{[r]}(z_1, \dots, z_n)$ is the set of decorated GKG s.t.:

- G is a decorated GKG of genus g and n white vertices, without square vertex;
- the white vertices are of degree 1, the variable of the face around the j^{th} one is z_j .

$$\mathcal{W}_{g,n}^{[r]}(z_1, \dots, z_n) = \left\{ \begin{array}{c} z_1 \\ \circ \text{---} \bullet \text{---} \text{[genus } g \text{ surface]} \text{---} \bullet \text{---} z_2 \text{---} \dots \text{---} \bullet \text{---} z_n \end{array} \right\}$$

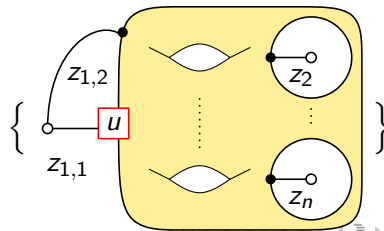
Square ciliated GKG:

$$\mathcal{U}_{g,n}^{[r]}(u; z_1, \dots, z_n) =$$



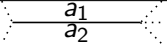



Multiciliated GKG:

$$\mathcal{S}_{g,n}^{[r]}(u; [z_{1,1}, z_{1,2}], z_2, \dots, z_n) =$$



Weight of a decorated GKG

Potential: $V(z) = \sum_{j=1}^{r+1} v_j \frac{z^j}{j}$

Object	Picture	Notation	Weight
Edge		$\mathcal{P}(a_1, a_2)$	$\frac{a_1 - a_2}{V'(a_1) - V'(a_2)}$
White vertex			1
Black vertex		$\mathcal{V}_k(a_1, \dots, a_k)$	$\mathcal{V}_1(a_1) = -V'(a_1)$ $\mathcal{V}_{k+1}(a_1, \dots, a_{k+1}) =$ $\frac{\mathcal{V}_k(a_1, \dots, a_k) - \mathcal{V}_k(a_1, \dots, a_{k-1}, a_{k+1})}{a_k - a_{k+1}}$
Square vertex		$Q_k(u; a_1, \dots, a_k)$	$\prod_{j=1}^k \frac{1}{u - a_j}$

The **weight** of G , a decorated GKG, is the product of the local weights:

$$w(G) = \prod_{\text{edges}} \mathcal{P}(a, b) \prod_{\text{black vertices}} \mathcal{V}_k(a_1, \dots, a_k) \prod_{\text{square vertices}} Q_m(u; b_1, \dots, b_m)$$

Remarks:

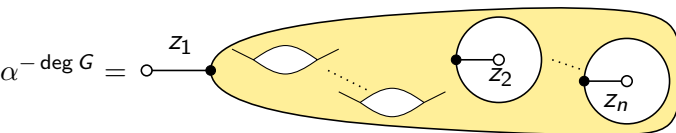
- due to the constraint on the black vertices, $\mathcal{V}_1(a_1)$ and $\mathcal{V}_2(a_1, a_2)$ are irrelevant;
- $\mathcal{V}_k(a_1, \dots, a_k) = \text{Res}_{u=\infty} Q_k(u; a_1, \dots, a_k) V'(u) du$

Generating series

Let G be a GKG. The **degree** of G is $\deg G = (r + 1)(2g - 2 + \#\text{faces})$.

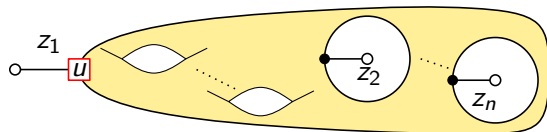
We define the main generating series:

$$W_{g,n}^{[r]}(z_1, \dots, z_n) = \sum_{G \in \mathcal{W}_{g,n}^{[r]}(z_1, \dots, z_n)} w(G) \alpha^{-\deg G} =$$

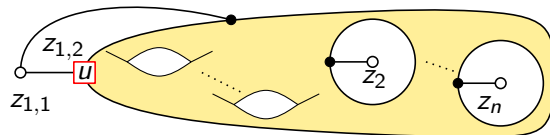


and the auxiliary ones:

$$U_{g,n}^{[r]}(u; z_1, \dots, z_n) =$$



$$S_{g,n}^{[r]}(u; [z_{1,1}, z_{1,2}], z_2, \dots, z_n) =$$



Remarks:

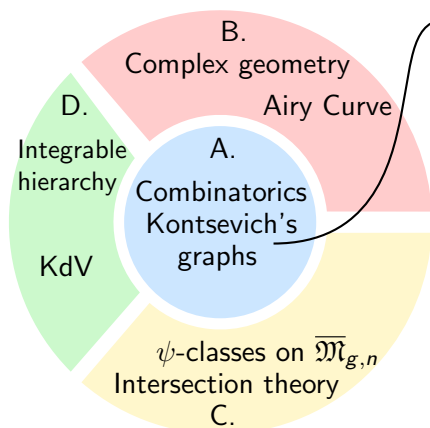
- They are well-defined formal power series in $\alpha^{-(r+1)}$: for a given degree, there is a finite number of graphs in each set.
- The automorphism factor of the graphs of those sets is 1.
- Those generating series are also called **correlation functions**.

Goal: compute $W_{g,n}^{[r]}(z_1, \dots, z_n)$ for all g, n .

Questions:

- **Why** this model, those weights, those generating functions?
- **How** to compute?

Origin of the model: inspiration from matrix models



Feynman graphs of the hermitian matrix model with external field:

$$Z = \int_{H_{N+\lambda}} dM e^{-N\text{Tr}\left(\frac{M^3}{3} - M\lambda^2\right)}$$

where $\lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$.

Natural generalisation: study the Feynman graphs of:

$$Z[r] = \int_{H_{N+\lambda}} dM e^{-N\text{Tr}(V(M) - M V'(\lambda))}, \quad V(z) = \sum_{j=1}^{r+1} v_j \frac{z^j}{j}$$

Carry out $M \leftarrow \tilde{M} + \lambda$ to erase the linear term:

$$Z[r] = C(\lambda) \int_{H_N} d\tilde{M} e^{-N \left(\frac{1}{2} \sum_{i,j=1}^N \tilde{M}_{i,j} \tilde{M}_{j,i} \frac{1}{\mathcal{P}(\lambda_i, \lambda_j)} - \sum_{\ell=3}^{r+1} \frac{1}{\ell} \sum_{i_1, \dots, i_{\ell}=1}^N \tilde{M}_{i_1, i_2} \tilde{M}_{i_2, i_3} \dots \tilde{M}_{i_{\ell}, i_1} \mathcal{V}_{\ell}(\lambda_{i_1}, \dots, \lambda_{i_{\ell}}) \right)}$$

Those correlation functions are the inspiration for the ciliated GKG:

$$\langle \tilde{M}_{i_1, i_1} \dots \tilde{M}_{i_n, i_n} \rangle = \frac{1}{Z[r]} \int_{H_{N+\lambda}} dM \tilde{M}_{i_1, i_1} \dots \tilde{M}_{i_n, i_n} e^{-N\text{Tr}(V(M) - M V'(\lambda))}$$

Combinatorial identities - 1

Notation: $\mathcal{R}(f) = \operatorname{Res}_{u=\infty} du V'(u)f(u)$.

Lemma (ciliated and square ciliated)

$$\frac{V'(z_1)}{V''(z_1)} \circlearrowleft_{z_1} = -\mathcal{R}\left(\left(u - z_1\right) \circlearrowleft_{z_1} \boxed{u}\right) - \delta_{g,0} \delta_{n,1} \frac{N}{V''(z_1)}$$

$$\frac{V'(z_1)}{V''(z_1)} W_{g,n}^{[r]}(z_1, \dots, z_n) = -\mathcal{R}\left(\left(u - z_1\right) U_{g,n}^{[r]}(u; z_1, \dots, z_n)\right) - \delta_{g,0} \delta_{n,1} \frac{N}{V''(z_1)}$$

Proof:

$$\mathcal{R}\left(\left(u - z_1\right) \circlearrowleft_{z_1} \boxed{u} \begin{array}{c} | a_1 \\ \vdots \\ | a_k \end{array}\right) = \frac{1}{V''(z_1)} \mathcal{R}\left(z_1 \begin{array}{c} | a_1 \\ \vdots \\ | a_k \end{array} \boxed{u}\right) = \frac{1}{V''(z_1)} z_1 \begin{array}{c} | a_1 \\ \vdots \\ | a_k \end{array}$$

Cancellation!

$$\mathcal{R}\left(\left(u - z_1\right) \circlearrowleft_{z_1} \boxed{u} a_1\right) = \frac{1}{V''(z_1)} \mathcal{R}\left(z_1 \begin{array}{c} | \\ \vdots \\ | \end{array} \boxed{u} a_1\right) = -\frac{1}{V''(z_1)} z_1 \begin{array}{c} | \\ \vdots \\ | \end{array} a_1$$

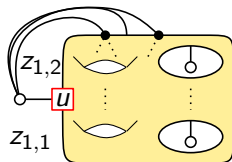
$$\mathcal{R}\left(\left(u - z_1\right) \circlearrowleft_{z_1} \boxed{u} \text{---}\right) = \frac{1}{V''(z_1)} \mathcal{R}\left(z_1 \begin{array}{c} | \\ \vdots \\ | \end{array} \boxed{u} \text{---}\right) = -\frac{V'(z_1)}{V''(z_1)} \circlearrowleft_{z_1}$$

Combinatorial identities - 2

Lemma (square ciliated and multiciliated)

$$\begin{aligned}
 & \left(\begin{array}{c} z_{1,2} \\ \vdots \\ z_{1,1} \end{array} \right) \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \\
 & = \frac{1}{\alpha^{r+1}} \frac{\mathcal{P}(z_{1,1}, z_{1,2})}{V'(z_{1,1}) - V'(z_{1,2})} \times \\
 & \left(V'''(z_{1,1}) \begin{array}{c} z_{1,1} \\ \vdots \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} - V'''(z_{1,2}) \begin{array}{c} z_{1,2} \\ \vdots \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right)
 \end{aligned}$$

Proof: Insert an edge $z_{1,1}|z_{1,2}$ at all possible places around the first marked face:



Then use:

$$\mathcal{V}_{k+1}(a_1, \dots, a_{k+1}) = \frac{\mathcal{V}_k(a_1, a_3, \dots, a_{k+1}) - \mathcal{V}_k(a_2, \dots, a_{k+1})}{a_1 - a_2}$$

to write the sum as a telescopic sum. Only the boundary terms remain!

Table of contents

- 1 Combinatorial model: definitions and preliminary identities
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- 2 Computing the generating series: Tutte's equation and Topological Recursion
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 - Topological recursion
- 3 Application: r -spin intersection numbers satisfy Topological Recursion

Tutte's recursion: principle

Goal: compute $W_{g,n}^{[r]}(z_1, \dots, z_n)$.

We use the – more convenient – auxiliary generating series $U_{g,n}^{[r]}$ to derive an equation “à la Tutte”.

(Combinatorial identity 1 allows to recover $W_{g,n}^{[r]}$ from $U_{g,n}^{[r]}$.)

$U_{g,n}^{[r]}$ is a formal power series in α :

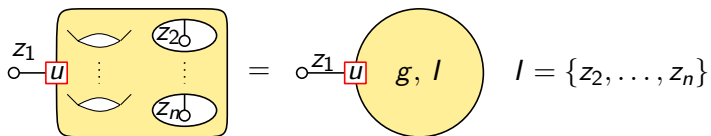
$$U_{g,n}^{[r]} = \sum_{d \geq 2g-2+n} \alpha^{-(r+1)d} U_{g,n}^{[r],d}$$

where $(U_{g,n}^{[r],d})_{d,g,n}$ is a 3-index family of rational functions, s.t.

$$g \geq 0, \quad n \geq 1, \quad d \geq 2g - 2 + n.$$

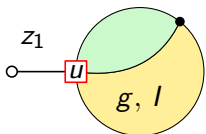
Tutte's equation is a recursive relation on $2g + n + d$ between the $U_{g,n}^{[r],d}$'s.

Notation:

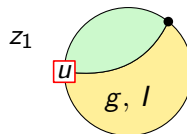


Procedure:

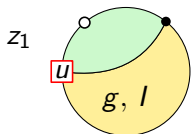
1. Start from a square ciliated GK



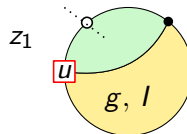
2. Erase the first cilium



3. Add a white vertex on the edge on the left

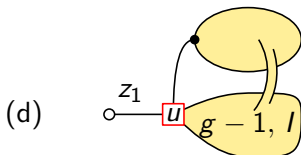
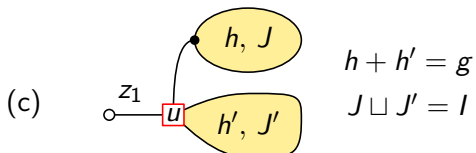
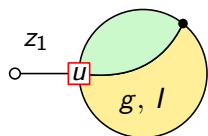
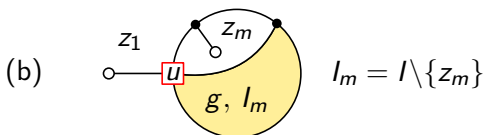
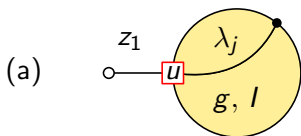


4. Cut the edge and look at the map we get



Tutte's equation

For $(g, n) \neq (0, 1)$, (the disc is special), there are 4 cases:



Tutte's equation – case (a)

$$\begin{aligned}
 & \text{Diagram 1} = \frac{\mathcal{P}(z_1, z_1)}{u - z_1} \text{Diagram 2} \\
 & \text{Combinatorial identity 2} \\
 & \left[\begin{aligned}
 & = \frac{\mathcal{P}(z_1, z_1)}{u - z_1} \frac{1}{\mathcal{P}(z_1, \lambda_j)} \text{Diagram 3} \\
 & = \frac{\mathcal{P}(z_1, z_1)}{u - z_1} \frac{1}{\alpha^{r+1}} \frac{1}{V'(z_1) - V'(\lambda_j)} \left(V''(z_1) \text{Diagram 4} - V''(\lambda_j) \text{Diagram 5} \right)
 \end{aligned} \right.
 \end{aligned}$$

Diagram 1: A yellow circle with a black dot at the top. A red square labeled 'u' is on the left edge. A line from 'u' goes to the dot. A label 'λ_j' is above the dot, and 'g, l' is below the circle. A label 'z₁' is to the left of the circle.

Diagram 2: A yellow circle with a black dot at the top. A red square labeled 'u' is on the left edge. A line from 'u' goes to the dot. A label 'λ_j' is above the dot, and 'g, l' is below the circle. A label 'z₁' is to the left of the circle.

Diagram 3: A yellow circle with a black dot at the top. A red square labeled 'u' is on the left edge. A line from 'u' goes to the dot. A label 'λ_j' is above the dot, and 'g, l' is below the circle. A label 'z₁' is to the left of the circle.

Diagram 4: A yellow circle with a red square labeled 'u' at the bottom. A line from 'u' goes to the bottom edge. A label 'g, l' is inside the circle. A label 'z₁' is to the left of the circle.

Diagram 5: A yellow circle with a red square labeled 'u' at the bottom. A line from 'u' goes to the bottom edge. A label 'g, l' is inside the circle. A label 'λ_j' is to the left of the circle.

This case gives the contribution:

$$\text{case (a)} = \frac{\alpha^{-(r+1)}}{u - z_1} \frac{1}{V''(z_1)} \sum_{j=1}^N \frac{V''(z_1) U_{g,n}^{[r]}(u; z_1, l) - V''(\lambda_j) U_{g,n}^{[r]}(u; \lambda_j, l)}{V'(z_1) - V(\lambda_j)}$$

Tutte's equation – case (b)

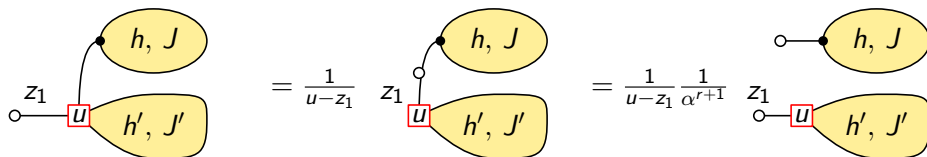
Combinatorial identity 3

$$\begin{aligned}
 & \left(\text{Diagram: Circle with points } z_1, z_m, \text{ and } u \text{ on the boundary. } z_1 \text{ and } z_m \text{ are connected by an arc. } u \text{ is connected to } z_1 \text{ by an arc. The interior is shaded yellow and labeled } g, l_m. \right) \\
 &= \frac{1}{V'''(z_m)} \frac{\partial}{\partial z_m} \left(\text{Diagram: Circle with points } z_1, z_m, \text{ and } u \text{ on the boundary. } z_1 \text{ and } z_m \text{ are connected by an arc. } u \text{ is connected to } z_m \text{ by an arc. The interior is shaded yellow and labeled } g, l_m. \right) \\
 &= \frac{\mathcal{P}(z_1, z_1)}{u - z_1} \frac{1}{\alpha^{r+1}} \frac{1}{V'''(z_m)} \frac{\partial}{\partial z_m} \frac{1}{V'(z_1) - V'(z_m)} \left(V''(z_1) \left(\text{Diagram: Circle with } z_1 \text{ and } u \text{ on the boundary, shaded yellow, labeled } g, l_m. \right) - V''(z_m) \left(\text{Diagram: Circle with } z_m \text{ and } u \text{ on the boundary, shaded yellow, labeled } g, l_m. \right) \right)
 \end{aligned}$$

This case gives the contribution:

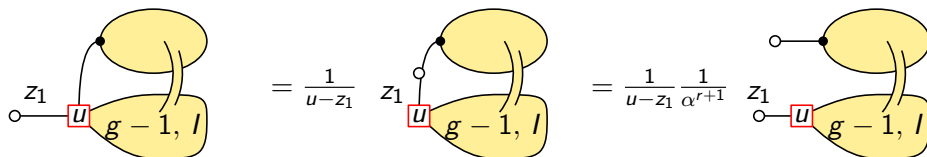
$$\text{case (b)} = \frac{\alpha^{-(r+1)}}{u - z_1} \frac{1}{V'''(z_1)} \sum_{m=2}^n \frac{1}{V'''(z_m)} \frac{\partial}{\partial z_m} \frac{V''(z_1) U_{g, n-1}^{[r]}(u; z_1, l_m) - V''(z_m) U_{g, n-1}^{[r]}(u; z_m, l_m)}{V'(z_1) - V'(z_m)}$$

Tutte's equation – cases (c) and (d)



This case gives the contribution:

$$\text{case (c)} = \frac{\alpha^{-(r+1)}}{u-z_1} \sum_{\substack{h+h'=g \\ J \sqcup J'=l}} W_{h,1+J}^{[r]}(z_1, J) U_{h',1+J'}^{[r]}(u; z_1, J')$$



which gives the contribution:

$$\text{case (d)} = \frac{\alpha^{-(r+1)}}{u-z_1} U_{g-1, n+1}^{[r]}(u; z_1, z_1, l)$$

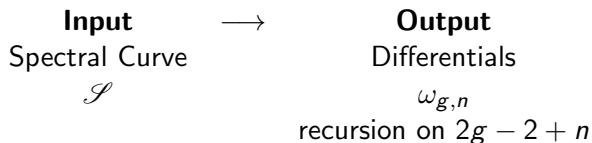
Tutte's equation for GK

For $(g, n) \neq (0, 1)$:

$$\begin{aligned}
 U_{g,n}^{[r]}(u; z_1, l) &= \frac{\alpha^{-(r+1)}}{u - z_1} \left(\frac{1}{V'''(z_1)} \sum_{j=1}^N \frac{V'''(z_1) U_{g,n}^{[r]}(u; z_1, l) - V'''(\lambda_j) U_{g,n}^{[r]}(u; \lambda_j, l)}{V'(z_1) - V(\lambda_j)} \right. \\
 &+ \frac{1}{V'''(z_1)} \sum_{m=2}^n \frac{1}{V'''(z_m)} \frac{\partial}{\partial z_m} \frac{V'''(z_1) U_{g,n-1}^{[r]}(u; z_1, l_m) - V'''(z_m) U_{g,n-1}^{[r]}(u; z_m, l_m)}{V'(z_1) - V(z_m)} \\
 &+ \sum_{\substack{h+h'=g \\ J \sqcup J'=l}} W_{h,1+J}^{[r]}(z_1, J) U_{h',1+J'}^{[r]}(u; z_1, J') \\
 &\left. + U_{g-1,n+1}^{[r]}(u; z_1, z_1, l) \right)
 \end{aligned}$$

Topological Recursion

Topological Recursion is a **procedure** developed by Chekhov–Eynard–Orantin ('07), see [Vincent Delecroix's talks](#).



Various applications:

- Combinatorics (Alexandrov–Mironov–Morozov '05)
- Gromov-Witten Invariants/Mirror symmetry (Bouchard–Klemm–Mariño–Pasquetti '08)
- Knot invariants (Borot–Eynard '12)
- Cohomological Field Theory (Dunin-Barkowski–Orantin–Shadrin–Spitz '12)
- Integrable hierarchies (Eynard–Garcia-Failde '19, Marchal–Orantin '19)
- Volumes of the moduli spaces (Eynard–Orantin '07)
- ...

More in detail...

Spectral curve:

$$\mathcal{S} = (\Sigma, \Sigma_0, x, y, \omega_{0,2})$$

where:

- Σ, Σ_0 are complex curves;
- $x, y : \Sigma \rightarrow \Sigma_0$ are coverings (x is branched);
- $\omega_{0,2} \in H^0(\Sigma^2, K_{\Sigma^2})$ is a symmetric bi-differential on Σ .

We note $\omega_{0,1}(z) = y(z)dx(z)$.

Recursion:

For $2g - 2 + n \geq 1$, the n -differential $\omega_{g,n}$ is given by:

$$\omega_{g,n}(z_1, I) = \operatorname{Res}_{\substack{z=a \\ dx(a)=0}} K_a(z_1, z) \left(\omega_{g-1, n+1}(z, \tau_a(z), I) + \sum_{\substack{h+h'=g \\ J \sqcup J' = I}} \omega_{h,1+J}(z, J) \omega_{h',1+J'}(\tau_a(z), J') \right)$$

where $I = \{z_2, \dots, z_n\}$, τ_a is the deck transformation around the branchpoint a , and K_a is the recursion's kernel.

Spectral curve for GKG

Definition

Let Q be the unique polynomial of degree r s.t.

$$Q(\zeta) \underset{\zeta \rightarrow \infty}{=} V' \left(\zeta + \alpha^{-(r+1)} \sum_{j=1}^N \frac{1}{Q'(\xi_j)(\zeta - \xi_j)} \right) + \mathcal{O}(1/\zeta)$$

where $Q(\xi_j) = V'(\lambda_j)$.

The coefficients of Q are formal power series in α .

Definition (Spectral curve for GKG)

$$\mathcal{S} = (\mathbb{P}^1, \mathbb{P}^1, x, y, \omega_{0,2}^{[r]})$$

with

- $x(\zeta) = Q(\zeta)$
- $y(\zeta) = \zeta + \alpha^{-(r+1)} \sum_{j=1}^N \frac{1}{Q'(\xi_j)(\zeta - \xi_j)}$, $\omega_{0,1}^{[r]}(\zeta) = \alpha^{r+1} y(\zeta) dx(\zeta)$
- $\omega_{0,2}^{[r]}(\zeta_1, \zeta_2) = \frac{d\zeta_1 \otimes d\zeta_2}{(\zeta_1 - \zeta_2)^2}$

Topological Recursion for GKG

Definition (Differentials for GKG)

Let ζ_j be the implicit function of z_j s.t.:

$$Q(\zeta_j) = V'(z_j), \quad \zeta_j \underset{z_j \rightarrow \infty}{=} z_j + \mathcal{O}(1).$$

For $2g - 2 + n \geq 1$, define the n differentials:

$$\omega_{g,n}^{[r]}(\zeta_1, \dots, \zeta_n) = W_{g,n}^{[r]}(z_1, \dots, z_n) dx(\zeta_1) \dots dx(\zeta_n).$$

Theorem (BCEG-F '21)

The $\omega_{g,n}^{[r]}(\zeta_1, \dots, \zeta_n)$'s satisfy topological recursion for the GKG spectral curve.

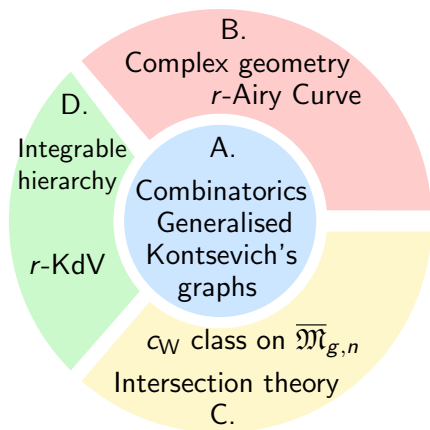
\Rightarrow we have a way to compute $W_{g,n}^{[r]}$ by recursion on $2g - 2 + n$.

The proof is rather technical and starts from Tutte's equation. It requires that the branchpoints of the spectral curve are simple: $Q'(\zeta)$ has simple roots (the potential and the λ 's are then called **generic**).

Table of contents

- 1 Combinatorial model: definitions and preliminary identities
 - Definition of the model
 - Origin of the model: matrix models
 - Combinatorial identities
- 2 Computing the generating series: Tutte's equation and Topological Recursion
 - Tutte's recursion for GKG
 - Topological recursion
- 3 Application: r -spin intersection numbers satisfy Topological Recursion

Witten's class



A. Generalised Kontsevich graphs:

Choose $V(z) = \frac{z^{r+1}}{r+1}$, $\lambda_1 = \dots = \lambda_N = \infty$.

B. r -Airy Curve:

$$\begin{aligned} \mathcal{S}^{[r]} &= (\Sigma, \Sigma_0, x(z), y(z), \omega_{0,2}(z_1, z_2)) \\ &= \left(\mathbb{P}^1, \mathbb{P}^1, z^r, z, \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2} \right) \end{aligned}$$

C. Witten's class / r -spin class: $c_W \in H^\bullet(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q})$.

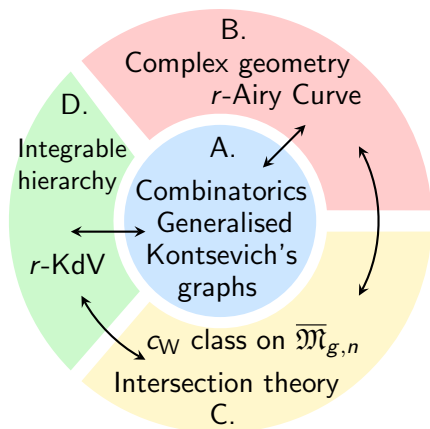
Intersection numbers:

$$\langle \tau_{d_1, a_1} \dots \tau_{d_n, a_n} \rangle_g = \int_{\overline{\mathfrak{M}}_{g,n}} c_W(a_1, \dots, a_n) \psi_1^{d_1} \dots \psi_n^{d_n}$$

$c_W(a_1, \dots, a_n)$ is defined from the moduli space of r -spin structures $\overline{\mathfrak{M}}_{g; a_1, \dots, a_n}^{1/r}$ (think of the points of $\overline{\mathfrak{M}}_{g; a_1, \dots, a_n}^{1/r}$ as r^{th} roots of canonical bundles over curves).

D. r -KdV hierarchy / r^{th} Gelfand-Dikii hierarchy, obtained as a reduction of the KdV hierarchy.

r -spin intersection numbers satisfy topological recursion



$B \leftrightarrow A$:

Main theorem + (Bouchard–Eynard '13): for $V(z) = \frac{z^{r+1}}{r+1}$ and $\lambda_j = \infty$, the $\omega_{g,n}^{[r]}$'s satisfy higher topological recursion with the r -Airy curve

$A \leftrightarrow D$:

The partition function of the matrix model

$$Z[r] = \int_{H_N} e^{-N\text{Tr}\left(\frac{M^{r+1}}{r+1} - M\lambda^r\right)}$$

is a tau-function of the r -KdV hierarchy satisfying the string equation (Adler–van Moerbeke '92)

$D \leftrightarrow C$:

The generating series of r -spin intersection numbers is a tau-function of the r -KdV hierarchy satisfying the string equation (conjecture: Witten '93, theorem: Faber–Shadrin–Zvonkine '10).

Corollary ($B \leftrightarrow A \leftrightarrow D \leftrightarrow C$ (Dunin-Barkowski–Norbury–Orantin–Popolitov–Shadrin '19))

The r -spin intersection numbers are computed via topological recursion with the r -Airy curve.

Conclusion

Also in the paper:

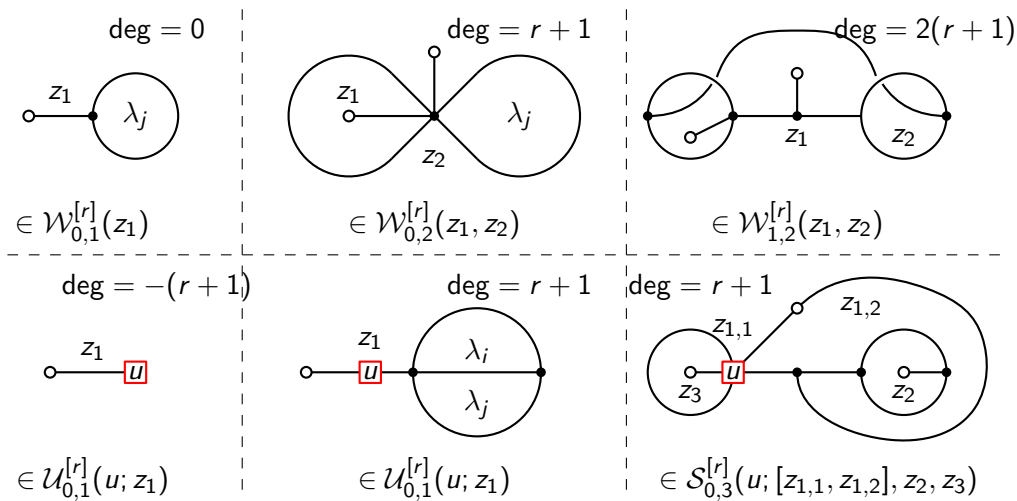
- The combinatorial model offers a wide range of spectral curves by tuning the potential V and the λ 's;
- In the proof of topological recursion for the model, we obtain a “combinatorial interpretation” for loop equations (coming from matrix models).

Work in progress (to appear soonish):

- Proof of a conjecture by Borot–Garcia-Failde: fully simple maps satisfy topological recursion, symplectic invariance of topological recursion;
- Study the intersection of Witten’s class by varying the spectral curve (DOSS formula).

Thank you for your attention!

Examples of GKG in various sets



Weights of black vertices

$$\mathcal{V}_1(a_1) = -V'(a_1)$$

$$\mathcal{V}_2(a_1, a_2) = -\frac{V'(a_1) - V'(a_2)}{a_1 - a_2} = -\frac{1}{\mathcal{P}(a_1, a_2)}$$

$$\mathcal{V}_3(a_1, a_2, a_3) = -\frac{V'(a_1)}{(a_1 - a_2)(a_1 - a_3)} - \frac{V'(a_2)}{(a_2 - a_1)(a_2 - a_3)} - \frac{V'(a_3)}{(a_3 - a_1)(a_3 - a_2)}$$

$$\mathcal{V}_k(a_1, \dots, a_k) = -\sum_{j=1}^k \frac{V'(a_j)}{\prod_{i \neq j} (a_j - a_i)}$$

Example: $V(z) = z^3/3$

$$\mathcal{V}_3(z_1, z_2, z_3) = -1, \quad \mathcal{V}_k(a_1, \dots, a_k) = 0, \quad k \geq 4.$$

Example: $V(z) = z^4/4$

$$\mathcal{V}_3(z_1, z_2, z_3) = -(z_1 + z_2 + z_3), \quad \mathcal{V}_4(z_1, \dots, z_4) = -1, \quad \mathcal{V}_k(a_1, \dots, a_k) = 0, \quad k \geq 5.$$

Tutte's equation for the disc

$$U_{0,1}^{[r]}(u; z_1) = \frac{\alpha^{-(r+1)}}{u - z_1} \frac{1}{V'''(z_1)} \sum_{j=1}^N \frac{V'''(z_1)U_{0,1}^{[r]}(u; z_1) - V'''(\lambda_j)U_{0,1}^{[r]}(u; \lambda_j)}{V'(z_1) - V'(\lambda_j)} \\ + \frac{\alpha^{-(r+1)}}{u - z_1} W_{0,1}^{[r]}(z_1)U_{0,1}^{[r]}(u; z_1) + \frac{\alpha^{r+1}}{(u - z_1)V'''(z_1)}$$

Disc and cylinder generating series

$$y(\zeta) = \zeta + \alpha^{-(r+1)} \sum_{j=1}^N \frac{1}{Q'(\xi_j)(\zeta - \xi_j)} = z + \alpha^{-(r+1)} \left(W_{0,1}^{[r]}(z) + \sum_{j=1}^N \frac{1}{V'(z) - V'(\lambda_j)} \right)$$

$$\omega_{0,2}^{[r]}(\zeta_1, \zeta_2) = \frac{d\zeta_1 d\zeta_2}{(\zeta_1 - \zeta_2)^2} = \left(W_{0,2}^{[r]}(z_1, z_2) + \frac{1}{(V'(z_1) - V'(z_2))^2} \right) V'''(z_1) dz_1 V'''(z_2) dz_2$$