Generalised Kontsevich graphs: topological recursion and r-spin intersection numbers

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Witten’s conjecture (’90): the generating series of intersection numbers of $\psi$-classes on the moduli spaces of stable curves

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g = \int_{\overline{M}_{g,n}} \psi_{d_1} \cdots \psi_{d_n}$$

is a tau-function of the KdV hierarchy (infinite set of non-linear differential equations).

Kontsevich (’92): generating functions of $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g = \text{volumes of the moduli spaces of Riemann surfaces} = \text{sums over weighted fatgraphs (Kontsevich’s graphs)}$.

Kontsevich (’92): the enumeration of the fatgraphs is solved by (what was later called) topological recursion on the Airy curve.

Dijkgraaf–Verlinde–Verlinde (’91): topological recursion on the Airy curve $\Leftrightarrow$ Virasoro constraints of the KdV hierarchy.

Theorem (Witten’s conjecture/Kontsevich’s theorem)

*The generating series of intersection numbers of $\psi$-classes on the moduli spaces of stable curves is a tau-function of the KdV hierarchy.*
**Question:** what is the complete picture if we consider Witten’s conjecture about $r$-spin classes and the $r$-KdV hierarchy ('93)?

Combinatorial side? Topological recursion side?
Plan

1. Combinatorial model: definitions and preliminary identities
   - Definition of the model
   - Origin of the model: matrix models
   - Combinatorial identities

2. Computing the generating series: Tutte’s equation and Topological Recursion
   - Tutte’s recursion for GKG
   - Topological recursion

3. Application: r-spin intersection numbers satisfy Topological Recursion
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Definition (Map)

- A **map** is a graph $G$ endowed with a cyclic order on the edges incident to each vertex.
- **Genus** $g$ of a connected map:

$$\#\text{vertices} - \#\text{edges} + \#\text{faces} = 2 - 2g$$

### Examples

- **#faces = 1**
  - $g = 1$
- **#faces = 3**
  - $g = 0$
Generalised Kontsevich Graphs

Let $r \geq 2$.

Definition (GKG)

A **Generalised Kontsevich Graph** (GKG) is a connected map which can carry 3 types of vertices:

- Black vertices, of degree $3 \leq d \leq r + 1$;
- Labelled white vertices (from 1 to $n$), of degree $d \geq 1$, with the constraints:
  1. each face is adjacent to at most one white vertex;
  2. if a white vertex and its half-edges are removed, the remaining half-edges – that where attached to the erased set – belong to the same face (**star constraint**);
- Square vertices, of degree $d \geq 1$. 
Decorated GKG

Faces adjacent to white vertices of a GKG are called **marked faces**, the other faces are **unmarked faces**.

**Definition (Decorated Generalised Kontsevich Graphs)**

Let $N \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$, and let $G$ be a GKG with $n$ white vertices. $G$ is a **decorated Generalised Kontsevich Graph** (decorated GKG) if:

- the marked faces around the $j^{th}$ white vertex carry the complex variables $z_{j,1}, \ldots, z_{j,k_j}$;
- each unmarked face carries a parameter belonging to $\{\lambda_1, \ldots, \lambda_N\}$;
- each square vertex carries the complex variable $u$. 

![Diagram of a decorated GKG](image)
Sets of interest

We are interested in the weighted enumeration of graphs of the following set:

Definition (ciliated GKG)

\( G \in \mathcal{W}_{g,n}^{[r]}(z_1, \ldots, z_n) \) is the set of decorated GKG s.t.:
- \( G \) is a decorated GKG of genus \( g \) and \( n \) white vertices, without square vertex;
- the white vertices are of degree 1, the variable of the face around the \( j^{th} \) one is \( z_j \).

\[
\mathcal{W}_{g,n}^{[r]}(z_1, \ldots, z_n) = \left\{ \begin{array}{c}
\end{array} \right.
\]

Square ciliated GKG:
\[ \mathcal{U}_{g,n}^{[r]}(u; z_1, \ldots, z_n) = \]

Multiciliated GKG:
\[ S_{g,n}^{[r]}(u; [z_{1,1}, z_{1,2}], z_2, \ldots, z_n) = \]
Weight of a decorated GKG

Potential: \( V(z) = \sum_{j=1}^{r+1} v_j z^j \)

<table>
<thead>
<tr>
<th>Object</th>
<th>Picture</th>
<th>Notation</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edge</td>
<td><img src="image" alt="Edge" /></td>
<td>( P(a_1, a_2) )</td>
<td>( \frac{a_1-a_2}{V'(a_1)-V'(a_2)} )</td>
</tr>
<tr>
<td>White vertex</td>
<td><img src="image" alt="White vertex" /></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>
| Black vertex | ![Black vertex](image)  | \( \mathcal{V}_k(a_1, \ldots, a_k) \) | \( \mathcal{V}_1(a_1) = -V'(a_1) \)
|              |                          |                 | \( \mathcal{V}_{k+1}(a_1, \ldots, a_{k+1}) = \frac{\mathcal{V}_k(a_1, \ldots, a_k) - \mathcal{V}_k(a_1, \ldots, a_{k-1}, a_{k+1})}{a_k-a_{k+1}} \) |
| Square vertex| ![Square vertex](image) | \( Q_k(u; a_1, \ldots, a_k) \) | \( \prod_{j=1}^{k} \frac{1}{u-a_j} \)                                  |

The weight of \( G \), a decorated GKG, is the product of the local weights:

\[
w(G) = \prod_{\text{edges}} P(a,b) \prod_{\text{black vertices}} \mathcal{V}_k(a_1, \ldots, a_k) \prod_{\text{square vertices}} Q_m(u; b_1, \ldots, b_m)
\]

Remarks:

- due to the constraint on the black vertices, \( \mathcal{V}_1(a_1) \) and \( \mathcal{V}_2(a_1, a_2) \) are irrelevant;
- \( \mathcal{V}_k(a_1, \ldots, a_k) = \text{Res}_{u=\infty} Q_k(u; a_1, \ldots, a_k) V'(u)\,du \)
Generating series

Let $G$ be a GKG. The **degree** of $G$ is $\text{deg } G = (r + 1)(2g - 2 + \#\text{faces})$.

We define the main generating series:

$$W_{g,n}^{[r]}(z_1, \ldots, z_n) = \sum_{G \in W_{g,n}^{[r]}(z_1, \ldots, z_n)} w(G) \alpha^{-\text{deg } G} = \sum_{G \in W_{g,n}^{[r]}(z_1, \ldots, z_n)} w(G) \alpha^{-\text{deg } G} =$$

and the auxiliary ones:

$$U_{g,n}^{[r]}(u; z_1, \ldots, z_n) =$$

$$S_{g,n}^{[r]}(u; [z_1, 1, z_1, 2], z_2, \ldots, z_n) =$$

**Remarks:**

- They are well-defined formal power series in $\alpha^{-(r+1)}$: for a given degree, there is a finite number of graphs in each set.
- The automorphism factor of the graphs of those sets is 1.
- Those generating series are also called **correlation functions**.
Goal: compute $W_{g,n}^r(z_1, \ldots, z_n)$ for all $g,n$.

Questions:

- **Why** this model, those weights, those generating functions?
- **How** to compute?
Origin of the model: inspiration from matrix models

Feynman graphs of the hermitian matrix model with external field:

\[ Z = \int_{H_N + \lambda} dM \ e^{-N \text{Tr} \left( \frac{M^3}{3} - M \lambda^2 \right)} \]

where \( \lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \).

**Natural generalisation:** study the Feynman graphs of:

\[ Z[r] = \int_{H_N + \lambda} dM \ e^{-N \text{Tr}(V(M) - M V'(\lambda))}, \quad V(z) = \sum_{j=1}^{r+1} v_j \frac{z^j}{j} \]

Carry out \( M \leftarrow \tilde{M} + \lambda \) to erase the linear term:

\[ Z[r] = C(\lambda) \int_{H_N} d\tilde{M} e^{-N \left( \frac{1}{2} \sum_{i,j=1}^{N} \tilde{M}_{i,j} \tilde{M}_{j,i} \frac{1}{P(\lambda_i, \lambda_j)} - \sum_{\ell=3}^{r+1} \frac{1}{\ell} \sum_{i_1, \ldots, i_\ell=1}^{N} \tilde{M}_{i_1, i_2} \tilde{M}_{i_2, i_3} \cdots \tilde{M}_{i_\ell, i_1} v_\ell(\lambda_{i_1}, \ldots, \lambda_{i_\ell}) \right)} \]

Those correlation functions are the inspiration for the ciliated GKG:

\[ \langle \tilde{M}_{i_1, i_1} \cdots \tilde{M}_{i_n, i_n} \rangle = \frac{1}{Z[r]} \int_{H_N + \lambda} dM \tilde{M}_{i_1, i_1} \cdots \tilde{M}_{i_n, i_n} e^{-N \text{Tr}(V(M) - M V'(\lambda))} \]
Combinatorial identities - 1

**Notation:** \( \mathcal{R}(f) = \text{Res}_{u=\infty} du \, V'(u)f(u) \).

**Lemma (ciliated and square ciliated)**

\[
\frac{V''(z_1)}{V'''(z_1)} W_{g,n}^{[r]}(z_1, \ldots, z_n) = -\mathcal{R} \left( (u - z_1) \frac{z_1}{a_1} \frac{1}{a_k} \right) - \delta_{g,0} \delta_{n,1} \frac{N}{V'''(z_1)}
\]

**Proof:**

\[
\mathcal{R} \left( (u - z_1) \frac{z_1}{a_1} \frac{1}{a_k} \right) = \frac{1}{V'''(z_1)} \mathcal{R} \left( z_1 \frac{a_1}{a_k} \right) = \frac{1}{V'''(z_1)} \mathcal{R} \left( z_1 \frac{a_1}{a_k} \right)
\]

Cancellation!

\[
\mathcal{R} \left( (u - z_1) \frac{z_1}{a_1} \right) = \frac{1}{V'''(z_1)} \mathcal{R} \left( z_1 \frac{a_1}{a_1} \right) = -\frac{1}{V'''(z_1)} \frac{z_1}{a_1}
\]

\[
\mathcal{R} \left( (u - z_1) \frac{z_1}{a_1} \right) = \frac{1}{V'''(z_1)} \mathcal{R} \left( z_1 \frac{z_1}{a_1} \right) = -\frac{V'(z_1)}{V'''(z_1)} \frac{z_1}{a_1}
\]
Lemma (square ciliated and multiciliated)

\[
\frac{1}{\alpha^{r+1}} \frac{P(z_{1,1}, z_{1,2})}{V'(z_{1,1}) - V''(z_{1,2})} \times
\]

\[
(V''(z_{1,1}) - V''(z_{1,2}))
\]

**Proof:** Insert an edge $z_{1,1}|z_{1,2}$ at all possible places around the first marked face:

Then use:

\[
\mathcal{V}_{k+1}(a_1, \ldots, a_{k+1}) = \frac{\mathcal{V}_k(a_1, a_3, \ldots, a_{k+1}) - \mathcal{V}_k(a_2, \ldots, a_{k+1})}{a_1 - a_2}
\]

to write the sum as a telescopic sum. Only the boundary terms remain!
Lemma (unciliated and ciliated)

Proof:

\[
\frac{1}{V''(z_j)} \frac{\partial}{\partial z_j} Z_j \bigg|_{a_1} = \frac{1}{V''(z_j)} \frac{\partial}{\partial z_j} Z_j \bigg|_{a_1}
\]
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3 Application: r-spin intersection numbers satisfy Topological Recursion
**Goal:** compute $W^{[r]}_{g,n}(z_1, \ldots, z_n)$.

We use the – more convenient – auxiliary generating series $U^{[r]}_{g,n}$ to derive an equation “à la Tutte”.

(Combinatorial identity 1 allows to recover $W^{[r]}_{g,n}$ from $U^{[r]}_{g,n}$.)

$U^{[r]}_{g,n}$ is a formal power series in $\alpha$:

$$U^{[r]}_{g,n} = \sum_{d \geq 2g - 2 + n} \alpha^{-(r+1)d} U^{[r],d}_{g,n}$$

where $(U^{[r],d}_{g,n})_{d,g,n}$ is a 3-index family of rational functions, s.t.

$$g \geq 0, \quad n \geq 1, \quad d \geq 2g - 2 + n.$$ 

Tutte’s equation is a recursive relation on $2g + n + d$ between the $U^{[r],d}_{g,n}$’s.
Notation:

\[ z_1 z_2 \ldots z_n = u g, I \]

\[ I = \{z_2, \ldots, z_n\} \]

Procedure:

1. Start from a square ciliated GKG

2. Erase the first cilium

3. Add a white vertex on the edge on the left

4. Cut the edge and look at the map we get
**Tutte’s equation**

For \((g, n) \neq (0, 1)\), (the disc is special), there are 4 cases:

(a) \[
\begin{align*}
\lambda_j \\
\end{align*}
\]

(b) \[
\begin{align*}
l_m = I \setminus \{z_m\}
\end{align*}
\]

(c) \[
\begin{align*}
h + h' &= g \\
J \sqcup J' &= I
\end{align*}
\]

(d) \[
\begin{align*}
g - 1, l
\end{align*}
\]
## Tutte’s equation – case (a)

\[
\begin{align*}
\frac{g, I_{\lambda_j}}{u - z_1} g, I & = \frac{P(z_1, z_1)}{u - z_1} \\
& = \frac{P(z_1, z_1)}{u - z_1} \frac{1}{P(z_1, \lambda_j)} 
\end{align*}
\]

**Combinatorial identity 2**

\[
\sum_{j=1}^{\infty} \frac{V''(z_1)}{V''(\lambda_j)} \left( V''(z_1) - V''(\lambda_j) \right)
\]

This case gives the contribution:

\[
\text{case (a)} = \alpha^{-(r+1)} \frac{1}{u - z_1} \sum_{j=1}^{\infty} \frac{V''(z_1) U_{g,n}^{[r]}(u; z_1, I) - V''(\lambda_j) U_{g,n}^{[r]}(u; \lambda_j, I)}{V'(z_1) - V(\lambda_j)}
\]
Tutte’s equation – case (b)

This case gives the contribution:

case (b) = \frac{\alpha^{-(r+1)}}{u - z_1} \sum_{m=2}^{n} \frac{1}{V''(z_1)} \frac{1}{V''(z_m)} \frac{\partial}{\partial z_m} \frac{V''(z_1) U_{g,n-1}^{[r]}(u; z_1, l_m) - V''(z_m) U_{g,n-1}^{[r]}(u; z_m, l_m)}{V'(z_1) - V(z_m)}
Tutte’s equation – cases (c) and (d)

\[
\begin{align*}
\text{(c) } &\quad \frac{1}{u - z_1} \sum_{h + h' = g, J \sqcup J' = l} W_{h,1}^{[r]}(z_1, J) U_{h',1}^{[r]}(u; z_1, J') \\
\text{(d) } &\quad \frac{1}{u - z_1} \frac{1}{\alpha^{r+1}} U_{g-1,n+1}^{[r]}(u; z_1, z_1, l)
\end{align*}
\]


**Tutte’s equation for GKG**

For \((g, n) \neq (0, 1)\):

\[
U_{g,n}^{[r]}(u; z_1, l) = \frac{\alpha^{-(r+1)}}{u - z_1} \left( \frac{1}{V''(z_1)} \sum_{j=1}^{N} V''(z_1)U_{g,n}^{[r]}(u; z_1, l) - V''(\lambda_j)U_{g,n}^{[r]}(u; \lambda_j, l) \right)
\]

\[
+ \frac{1}{V''(z_1)} \sum_{m=2}^{n} \frac{1}{V''(z_m)} \frac{\partial}{\partial z_m} \left( V''(z_1)U_{g,n-1}^{[r]}(u; z_1, l) - V''(z_m)U_{g,n-1}^{[r]}(u; z_m, l) \right)
\]

\[
+ \sum_{h+h'=g \atop J \sqcup J'=l} W_{h,1,J}(z_1, J) U_{h',1,J'}^{[r]}(u; z_1, J')
\]

\[+ U_{g-1,n+1}^{[r]}(u; z_1, z_1, l) \]
Topological Recursion

Topological Recursion is a **procedure** developed by Chekhov–Eynard–Orantin ('07), see Vincent Delecroix’s talks.

\[
\text{Input} \quad \xrightarrow{\text{recursion on } 2g - 2 + n} \quad \text{Output}
\]

Spectral Curve \( \mathcal{S} \) \quad \omega_{g,n} \quad Differentials

Various applications:

- Combinatorics (Alexandrov–Mironov–Morozov ’05)
- Gromov-Witten Invariants/Mirror symmetry (Bouchard–Klemm–Mariño–Pasquetti ’08)
- Knot invariants (Borot–Eynard ’12)
- Cohomological Field Theory (Dunin-Barkowski–Orantin–Shadrin–Spitz ’12)
- Integrable hierarchies (Eynard–Garcia-Failde ’19, Marchal–Orantin ’19)
- Volumes of the moduli spaces (Eynard–Orantin ’07)
- ...
More in detail...

Spectral curve:

$$\mathcal{S} = (\Sigma, \Sigma_0, x, y, \omega_{0,2})$$

where:

- $\Sigma, \Sigma_0$ are complex curves;
- $x, y : \Sigma \to \Sigma_0$ are coverings ($x$ is branched);
- $\omega_{0,2} \in H^0(\Sigma^2, K_{\Sigma}^{\otimes 2})$ is a symmetric bi-differential on $\Sigma$.

We note $\omega_{0,1}(z) = y(z)dx(z)$.

Recursion:

For $2g - 2 + n \geq 1$, the $n$-differential $\omega_{g,n}$ is given by:

$$\omega_{g,n}(z_1, I) = \operatorname{Res}_{z=a} K_a(z_1, z) \left( \omega_{g-1,n+1}(z, \tau_a(z), I) + \sum_{h+h'=g, \exists J, J' \subseteq I} \omega_{h,1+J}(z, J) \omega_{h',1+J'}(\tau_a(z), J') \right)$$

where $I = \{z_2, \ldots, z_n\}$, $\tau_a$ is the deck transformation around the branchpoint $a$, and $K_a$ is the recursion’s kernel.
Spectral curve for GKG

Definition

Let \( Q \) be the unique polynomial of degree \( r \) s.t.

\[
Q(\zeta) = V'(\zeta + \alpha^{-(r+1)} \sum_{j=1}^{N} \frac{1}{Q'(\xi_j)(\zeta - \xi_j)}) + O(1/\zeta)
\]

where \( Q(\xi_j) = V'(\lambda_j) \).

The coefficients of \( Q \) are formal power series in \( \alpha \).

Definition (Spectral curve for GKG)

\[
\mathcal{S} = (\mathbb{P}^1, \mathbb{P}^1, x, y, \omega^{[r]}_{0,2})
\]

with

- \( x(\zeta) = Q(\zeta) \)
- \( y(\zeta) = \zeta + \alpha^{-(r+1)} \sum_{j=1}^{N} \frac{1}{Q'(\xi_j)(\zeta - \xi_j)} \), \( \omega^{[r]}_{0,1}(\zeta) = \alpha^{r+1}y(\zeta)d\zeta(\zeta) \)
- \( \omega^{[r]}_{0,2}(\zeta_1, \zeta_2) = \frac{d\zeta_1 \otimes d\zeta_2}{(\zeta_1 - \zeta_2)^2} \)
Topological Recursion for GKG

Definition (Differentials for GKG)
Let \( \zeta_j \) be the implicit function of \( z_j \) s.t.:
\[
Q(\zeta_j) = V'(z_j), \quad \zeta_j \xrightarrow{z_j \to \infty} z_j + O(1).
\]

For \( 2g - 2 + n \geq 1 \), define the \( n \) differentials:
\[
\omega^{[r]}_{g,n}(\zeta_1, \ldots, \zeta_n) = W^{[r]}_{g,n}(z_1, \ldots, z_n) \, dx(\zeta_1) \ldots dx(\zeta_n).
\]

Theorem (BCEG-F '21)
The \( \omega^{[r]}_{g,n}(\zeta_1, \ldots, \zeta_n) \)'s satisfy topological recursion for the GKG spectral curve.

\( \Rightarrow \) we have a way to compute \( W^{[r]}_{g,n} \) by recursion on \( 2g - 2 + n \).
The proof is rather technical and starts from Tutte’s equation. It requires that the branchpoints of the spectral curve are simple: \( Q'(\zeta) \) has simple roots (the potential and the \( \lambda \)'s are then called generic).
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**Witten’s class**

A. Generalised Kontsevich graphs:
Choose \( V(z) = \frac{z^{r+1}}{r+1}, \lambda_1 = \cdots = \lambda_N = \infty. \)

B. \( r \)-Airy Curve:

\[
\mathcal{J}[r] = (\Sigma, \Sigma_0, x(z), y(z), \omega_{0,2}(z_1, z_2)) = \left( \mathbb{P}^1, \mathbb{P}^1, z^r, z, \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2} \right)
\]

C. Witten’s class / \( r \)-spin class: \( c_W \in H^\bullet(\overline{M}_{g,n}, \mathbb{Q}). \)

Intersection numbers:

\[
\langle \tau_{d_1,a_1} \cdots \tau_{d_n,a_n} \rangle_g = \int_{\overline{M}_{g,n}} c_W(a_1, \ldots, a_n) \psi_1^{d_1} \cdots \psi_n^{d_n}
\]

\( c_W(a_1, \ldots, a_n) \) is defined from the moduli space of \( r \)-spin structures \( \overline{M}_{1/r}^{1/r}_{g;a_1,\ldots,a_n} \) (think of the points of \( \overline{M}_{1/r}^{1/r}_{g;a_1,\ldots,a_n} \) as \( r \)th roots of canonical bundles over curves).

D. \( r \)-KdV hierarchy / \( r \)th Gelfand-Dikii hierarchy, obtained as a reduction of the KdV hierarchy.
r-spin intersection numbers satisfy topological recursion

**B ↔ A:**
Main theorem + (Bouchard–Eynard ’13): for $V(z) = \frac{z^{r+1}}{r+1}$ and $\lambda_j = \infty$, the $\omega_{g,n}^{[r]}$’s satisfy higher topological recursion with the $r$-Airy curve

**A ↔ D:**
The partition function of the matrix model

$$Z[r] = \int_{H_N} e^{-N \text{Tr} \left( \frac{M^{r+1}}{r+1} - M \lambda^r \right)}$$

is a tau-function of the $r$-KdV hierarchy satisfying the string equation (Adler–van Moerbeke ’92)

**D ↔ C:**
The generating series of $r$-spin intersection numbers is a tau-function of the $r$-KdV hierarchy satisfying the string equation (conjecture: Witten ’93, theorem: Faber–Shadrin–Zvonkine ’10).

**Corollary (B ↔ A ↔ D ↔ C** (Dunin-Barkowski–Norbury–Orantin–Polopitov–Shadrin ’19))

*The r-spin intersection numbers are computed via topological recursion with the r-Airy curve.*
Conclusion

Also in the paper:

- The combinatorial model offers a wide range of spectral curves by tuning the potential $V$ and the $\lambda$’s;
- In the proof of topological recursion for the model, we obtain a “combinatorial interpretation” for loop equations (coming from matrix models).

Work in progress (to appear soonish):

- Proof of a conjecture by Borot–Garcia-Failde: fully simple maps satisfy topological recursion, symplectic invariance of topological recursion;
- Study the intersection of Witten’s class by varying the spectral curve (DOSS formula).
Thank you for your attention!
Examples of GKG in various sets

- \( \lambda_j \in \mathcal{W}_{0,1}^{[r]}(z_1) \) with degree \( 0 \)
- \( \lambda_j \in \mathcal{W}_{0,2}^{[r]}(z_1, z_2) \) with degree \( r + 1 \)
- \( \lambda_j \in \mathcal{W}_{1,2}^{[r]}(z_1, z_2) \) with degree \( 2(r + 1) \)

- \( z_1 \in U_{0,1}^{[r]}(u; z_1) \) with degree \( -(r + 1) \)
- \( \lambda_i \in U_{0,1}^{[r]}(u; z_1) \) with degree \( r + 1 \)
- \( z_1 \in S_{0,3}^{[r]}(u; [z_{1,1}, z_{1,2}], z_2, z_3) \) with degree \( r + 1 \)
Weights of black vertices

\[ V_1(a_1) = -V'(a_1) \]
\[ V_2(a_1, a_2) = -\frac{V'(a_1) - V'(a_2)}{a_1 - a_2} = -\frac{1}{P(a_1, a_2)} \]
\[ V_3(a_1, a_2, a_3) = -\frac{V'(a_1)}{(a_1 - a_2)(a_1 - a_3)} - \frac{V'(a_2)}{(a_2 - a_1)(a_2 - a_3)} - \frac{V'(a_3)}{(a_3 - a_1)(a_3 - a_2)} \]
\[ V_k(a_1, \ldots, a_k) = -\sum_{j=1}^{k} \frac{V'(a_j)}{\prod_{i \neq j}(a_j - a_i)} \]

Example: \( V(z) = z^3/3 \)

\[ V_3(z_1, z_2, z_3) = -1, \quad V_k(a_1, \ldots, a_k) = 0, \quad k \geq 4. \]

Example: \( V(z) = z^4/4 \)

\[ V_3(z_1, z_2, z_3) = -(z_1 + z_2 + z_3), \quad V_4(z_1, \ldots, z_4) = -1, \quad V_k(a_1, \ldots, a_k) = 0, \quad k \geq 5. \]
Tutte’s equation for the disc

\[ U_{r}^{[r]}(u; z_1) = \frac{\alpha^{-(r+1)}}{u - z_1} \frac{1}{V''(z_1)} \sum_{j=1}^{N} \frac{V''(z_1) U_{0,1}^{[r]}(u; z_1) - V''(\lambda_j) U_{0,1}^{[r]}(u; \lambda_j)}{V'(z_1) - V'(\lambda_j)} \]

\[ + \frac{\alpha^{-(r+1)}}{u - z_1} W_{0,1}^{[r]}(z_1) U_{0,1}^{[r]}(u; z_1) + \frac{\alpha^{r+1}}{(u - z_1)V''(z_1)} \]
Disc and cylinder generating series

\[
y(\zeta) = \zeta + \alpha^{-(r+1)} \sum_{j=1}^{N} \frac{1}{Q'(\xi_j)(\zeta - \xi_j)} = z + \alpha^{-(r+1)} \left( W_{0,1}^{[r]}(z) + \sum_{j=1}^{N} \frac{1}{V'(z) - V'(\lambda_j)} \right)
\]

\[
\omega_{0,2}^{[r]}(\zeta_1, \zeta_2) = \frac{d\zeta_1 d\zeta_2}{(\zeta_1 - \zeta_2)^2} = \left( W_{0,2}^{[r]}(z_1, z_2) + \frac{1}{(V'(z_1) - V'(z_2))^2} \right) V''(z_1)dz_1 V''(z_2)dz_2
\]