Théorèmes arithmétiques démontrés par Théorie des Langages

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À la mémoire de Maurice Nivat (1937-2017)

≪ à chaque instant de notre vie consciente nous mettons en œuvre un algorithme ≫

Maurice Nivat
(Interview sur
l'option informatique,
Didapro [1], 2011)

1. https://www.youtube.com/watch?v=y-2zK7TxUyI

Motivations

To extend Chomsky hierarchy to subsets of positive integers. To prove elementary number-theoretical results by means of language theory.

Part I
Kassel-Reutenauer polynomials

Computational definition of $P_n(q)$

Definition

For any integer $n \geq 1$, we define the nth *Kassel-Reutenauer polynomial* as follows :

$$P_n(q) := a_{n,0} + \sum_{k=1}^{n-1} a_{n,k} \left(q^{n-1+k} + q^{n-1-k} \right),$$

where $a_{n,k} := \# \left\{ d | n : \quad d - 2\frac{n}{d} \leq 2k < 2d - \frac{n}{d} \right\}$. Furthermore, we define $C_n(q) := (q-1)^2 P_n(q)$.

Generating function

Theorem

$$\prod_{m=1}^{\infty} \frac{(1-t^m)^2}{(1-qt^m)(1-q^{-1}t^m)} = 1 + \sum_{n=1}^{\infty} \frac{C_n(q)}{q^n} t^n.$$

This result is essentially identity (9.2) in Nathan Jacob Fine, "Basic hypergeometric series and applications", No. 27, American Mathematical Soc., 1988.

Complex geometric interpretation of $C_n(q)$

Consider the Hilbert scheme $H^n_{\mathbb{C}} := \mathrm{Hilb}^n\left(\left(\mathbb{A}^1_{\mathbb{C}}\backslash\{0\}\right)\times\left(\mathbb{A}^1_{\mathbb{C}}\backslash\{0\}\right)\right)$ of n points on the bidimensional complex torus.

Theorem (Hausel, Letellier and Rodriguez-Villegas, 2011)

For each $n \geq 1$, the E-polynomial of $H^n_{\mathbb{C}}$ is $C_n(q)$.

Complex geometric interpretation of $P_n(q)$

Notice that $(\mathbb{C}\backslash\{0\}) \times (\mathbb{C}\backslash\{0\})$ acts naturally on

$$\left(\mathbb{A}^1_{\mathbb{C}}\backslash\{0\}\right) \times \left(\mathbb{A}^1_{\mathbb{C}}\backslash\{0\}\right).$$

This action induces an action of $(\mathbb{C}\backslash\{0\}) \times (\mathbb{C}\backslash\{0\})$ on $H^n_{\mathbb{C}}$. Define the geometric quotient $\widetilde{H}^n_{\mathbb{C}} := H^n_{\mathbb{C}}//\left((\mathbb{C}\backslash\{0\}) \times (\mathbb{C}\backslash\{0\})\right)$.

Theorem (Hausel, Letellier and Rodriguez-Villegas, 2011)

For each $n \geq 1$, the E-polynomial of $\widetilde{H}^n_{\mathbb{C}}$ is $P_n(q)$.

Evaluation at roots of unity

Theorem (Kassel and Reutenauer, 2016)

For any integer $n \geq 1$,

(i) $P_n(1)$ *is the sum of divisors of* n,

(ii) $C_n(-1)$ *is the number of integer solutions to the equation* $x^2 + y^2 = n$,

(iii) $\left|C_n\left(\sqrt{-1}\right)\right|$ *is the number of integer solutions to the equation* $x^2 + 2y^2 = n$,

(iv) $6Re\ P_n\left(\frac{-1+\sqrt{-3}}{2}\right)$ *is the number of integer solutions to the equation* $x^2 + xy + y^2 = n$.

Finite fields geometric interpretation of $C_n(q)$

Consider the Hilbert scheme

$$H^n_{\mathbb{F}_q} := \mathrm{Hilb}^n\left(\left(\mathbb{A}^1_{\mathbb{F}_q}\backslash\{0\}\right)\times\left(\mathbb{A}^1_{\mathbb{F}_q}\backslash\{0\}\right)\right)$$

of n points on the bidimensional $\mathbb{F}_q$-torus.

Theorem (Kassel and Reutenauer, 2015)

For each $n \geq 1$,

$$\sum_{m=1}^{\infty} C_n\left(q^m\right) t^m = \frac{t\frac{d}{dt} Z_{H^n_{\mathbb{F}_q}}(t)}{Z_{H^n_{\mathbb{F}_q}}(t)},$$

where $Z_{H^n_{\mathbb{F}_q}}(t)$ is the local zeta function of $H^n_{\mathbb{F}_q}$.

Evaluation at prime powers

Theorem (Kassel and Reutenauer, 2015)

For any prime power q and any integer $n \geq 1$, $C_n(q)$ is the number of ideals I of the group algebra $\mathbb{F}_q[\mathbb{Z} \oplus \mathbb{Z}]$ such that $\mathbb{F}_q[\mathbb{Z} \oplus \mathbb{Z}]/I$ is a vector space of dimension n over $\mathbb{F}_q$.

The coefficients of $P_n(q)$

Theorem (J. M. R. C., 2017)

For any integer $n \geq 1$,

(i) the largest coefficient of $P_n(q)$ is $F(n)$, where $F(n)$ is the Erdös-Nicolas function[a], i.e.

$$F(n) := \max_{t>0} \#\{d|n : \quad t < d \leq 2t\}.$$

(ii) the polynomial $P_n(q)$ has a coefficient greater than 1 if and only if $2n$ is the perimeter[b] of a Pythagorean triangle,

(iii) all the coefficients of $P_n(q)$ are non-zero if and only if n is 2-densely divisible[c].

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a. Paul Erdös, Jean-Louis Nicolas. Méthodes probabilistes et combinatoires en théorie des nombres. Bull. SC. Math **2** (1976) : 301–320.

b. The perimeter of a Pythagorean triangle is always an even integer.

c. i.e. the quotient of two consecutive divisors of n is less than or equal to 2. Densely divisible numbers were introduced by the international team *polymath8*, led by Terence Tao, in order to improve Zhang's bounded gaps between primes.

Odd-trapezoidal numbers

An integer $n \geq 1$ is an *odd-trapezoidal number* if for each pair of integers $a \geq 1$ and $k \geq 1$, the equality

$$n = a + (a+1) + (a+2) + ... + (a+k-1)$$

implies that k is odd.

Odd-trapezoidal numbers

Theorem (J. M. R. C., 2017)

For any integer $n \geq 1$, we have that n is odd-trapezoidal if and only if

$$a_{n,0} \geq a_{n,1} \geq a_{n,2} \geq \ldots \geq a_{n,n-1},$$

where $a_{n,0}, a_{n,1}, a_{n,2}, \ldots, a_{n,n-1}$ are the coefficients in the computational definition of $P_n(q)$.

Conclusion of Part I

From the polynomial $P_n(q)$ it is computationally easy to derive the following information about n :

(i) Whether or not $2n$ is the perimeter of a Pythagorean triangle.

(ii) Whether or not n is 2-densely divisible.

(iii) Whether or not n is odd-trapezoidal.

(iv) The number of middle divisors [2] of n.

(v) The number of integer solutions to the equations $x^2 + y^2 = n$, $x^2 + 2y^2 = n$ and $x^2 + xy + y^2 = n$.

(vi) The number of ideals I of the group algebra $\mathbb{F}_q[\mathbb{Z} \oplus \mathbb{Z}]$ such that $\mathbb{F}_q[\mathbb{Z} \oplus \mathbb{Z}]/I$ is a vector space of dimension n over $\mathbb{F}_q$.

(vii) The value of Erdös-Nicolas function at n.

(viii) The sum of divisors of n.

(ix) Topological information about $H^n_{\mathbb{C}}$ and $\widetilde{H}^n_{\mathbb{C}}$.

2. i.e the divisors d satisfying $\sqrt{\frac{n}{2}} < d \leq \sqrt{2n}$.

Part II
Language Theory

Passage from Part I to Part II

(i) We will encode part of the information from Kassel-Reutenauer polynomials into formal words.

(ii) We will translate some of the properties satisfied by Kassel-Reutenauer polynomials into language-theoretical statements.

The non-zero coefficients of $C_n(q)$

We can express $C_n(q) \in \mathbb{Z}[q]$, in a unique way, as follows [3],

$$C_n(q) = \eta_0 q^{e_0} + \eta_1 q^{e_1} + ... + \eta_k q^{e_k},$$

for some $\eta_0, \eta_1, ..., \eta_k \in \{+1, -1\}$ and $e_0, e_1, ..., e_k \in \mathbb{Z}$ satisfying :

(i) $e_0 \geq e_1 \geq ... \geq e_k \geq 0$,

(ii) for any $0 \leq j \leq k-1$, if $e_j = e_{j+1}$, then $\eta_j = \eta_{j+1}$.

So, the vector $\mathrm{KR}(n) := (\eta_0, \eta_1, ..., \eta_k) \in \{+1, -1\}^{k+1}$ is well-defined. By abuse of notation, we will write $\mathrm{KR}(n)$ as a word over the alphabet $\{+, -\}$, identifying $+ \leftrightarrow +1$ and $- \leftrightarrow -1$.

3. Notice that each positive coefficient of $C_n(q)$ corresponds to the multiplicity of a pole of $Z_{H^n_{\mathbb{F}_q}}(t)$. Similarly, each negative coefficient of $C_n(q)$ corresponds to the multiplicity of a zero of the same rational function.

Example

For $n = 6$,

$$\begin{aligned} C_6(q) &= q^{12} - q^{11} + q^7 - 2q^6 + q^5 - q + 1 \\ &= +q^{12} - q^{11} + q^7 - q^6 - q^6 + q^5 - q + 1. \end{aligned}$$

Therefore,

$$\mathrm{KR}(6) = + \ - \ + \ - \ - \ + \ - \ + \, .$$

A hidden pattern

Notice that

$$\mathrm{KR}(75) = + - - + + - + - - + - + + - + - - + - + + - -+$$

can be obtained from the well-matched parentheses

$$(\,)(\,(\,)\,)(\,(\,)\,)(\,)$$

by means of the letter-by-letter substitution [4]

$$\begin{aligned} \mu : \{(,)\}^* &\longrightarrow \{+,-\}^*, \\ (&\mapsto +-, \\ ) &\mapsto -+ . \end{aligned}$$

$$\mathrm{KR}(75) = \underbrace{+-}_{(}\underbrace{-+}_{)}\underbrace{+-}_{(}\underbrace{+-}_{(}\underbrace{-+}_{)}\underbrace{-+}_{)}\underbrace{+-}_{(}\underbrace{+-}_{(}\underbrace{-+}_{)}\underbrace{-+}_{)}\underbrace{+-}_{(}\underbrace{-+}_{)}$$

4. i.e. morphism of free monoids. Here, Σ^* denotes the free monoid over the alphabet Σ.

Well-matched parentheses

It follows from the computational definition of $C_n(q)$ that $\mathrm{KR}(n)$ is palindromic. The following property is less obvious.

Theorem (J. M. R. C., 2017)

For each integer $n \geq 1$, $\mathrm{KR}(n) = \mu(w_n)$, for some well-matched parentheses w_n.

Define the function $\delta : \mathbb{Z}_{\geq 1} \longrightarrow \{(,)\}^*$ by $\delta(n) := w_n$.

A direct computation of $\delta(n)$

The following result can be interpreted as a language-theoretical version of a formula for the coefficients of $C_n(q)$ due to Kassel and Reutenauer.

Theorem (J. M. R. C., 2017)

Let $n \geq 1$ be an integer. Denote D_n the set of divisors of n. Define $2D_n := \{2d : \quad d \in D_n\}$. Let $\tau_1 < \tau_2 < ... < \tau_k$ be the elements of $D_n \triangle 2D_n$ written in increasing order. Then,

$$\delta(n) = t_1\, t_2 \ldots t_k,$$

where

$$t_i := \begin{cases} (& \text{if } \tau_i \in D_n \setminus (2D_n), \\ ) & \text{if } \tau_i \in (2D_n) \setminus D_n. \end{cases}$$

Sets and languages

Definition

Let Σ be a finite alphabet. Given a set $S \subseteq \mathbb{Z}_{\geq 1}$, we say that S is *rational* (*context-free*) with respect to a function $f : \mathbb{Z}_{\geq 1} \longrightarrow \Sigma^*$, if

$$S = f^{-1}(L)$$

for some rational (context-free) language $L \subseteq \Sigma^*$.

Rational sets with respect to δ

Theorem (J. M. R. C., 2017)

The following sets are rational with respect to δ,

(i) the empty set of integers,

(ii) all the integers,

(iii) powers of 2,

(iv) semi-perimeters[a] of Pythagorean triangles.

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a. The semi-perimeter is a half of the perimeter.

Blocks

The *number of blocks* of an integer $n \geq 1$ is defined as the number of connected components of the topological space

$$\bigcup_{d|n} [d, 2d].$$

Notice that n is 2-densely divisible if and only if n has only one block.

Context-free sets with respect to δ

Theorem (J. M. R. C., 2017)

The following sets are context-free with respect to δ,

(i) integer having exactly k blocks, for any fixed $k \geq 1$,

(ii) numbers n satisfying $F(n) \geq h$, for any fixed integer $h \geq 1$, where $F(n)$ is the Erdös-Nicolas function.

Definition of $\widehat{\delta}$

For all $n \geq 1$,

$$\widehat{\delta}(n) := \psi\, \delta\, (n)\,,$$

where $\psi : \{((,)),(),)(\}^* \longrightarrow \{A, B, C, D\}^*$ satisfies, for all $w \in \{((,)),(),)(\}^*$,

$$\begin{aligned} \psi\varepsilon &:= \varepsilon, \\ \psi(w) &:= A\psi w, \\ \psi)w(&:= B\psi w, \\ \psi(w(&:= C\psi w, \\ \psi)w) &:= D\psi w. \end{aligned}$$

Hirschhorn function

We define the *Hirschhorn function*[5], $\mathrm{H} : \mathbb{Z}_{\geq 1} \times \{0,1\} \longrightarrow \mathbb{Z}_{\geq 0}$ by means of the expression

$$\mathrm{H}(n,b) := \#\left\{(a,k) \in \Pi_n : \quad k \equiv b \pmod 2\right\},$$

where Π_n is the set of pairs $(a,k) \in (\mathbb{Z}_{\geq 1})^2$ such that

$$n = a + (a+1) + (a+2) + ... + (a+k-1).$$

Notice that $\mathrm{H}(n,b) = 0$ if and only if n is odd-trapezoidal.

5. M. D. Hirschhorn and P. M. Hirschhorn. "Partitions into consecutive parts." (2009).

Rational sets with respect to $\hat{\delta}$

Theorem (J. M. R. C., 2017)

Given $k \in \mathbb{Z}_{\geq 0}$ and $b \in \{0, 1\}$, the set of integers $n \geq 1$ satisfying $\mathrm{H}(n, b) \geq k$ is rational with respect to $\hat{\delta}$.

Context-free sets with respect to $\hat{\delta}$

Theorem (J. M. R. C., 2017)

For each integer $k \geq 1$, the set of numbers n having exactly k middle divisors is context-free with respect to $\hat{\delta}$.

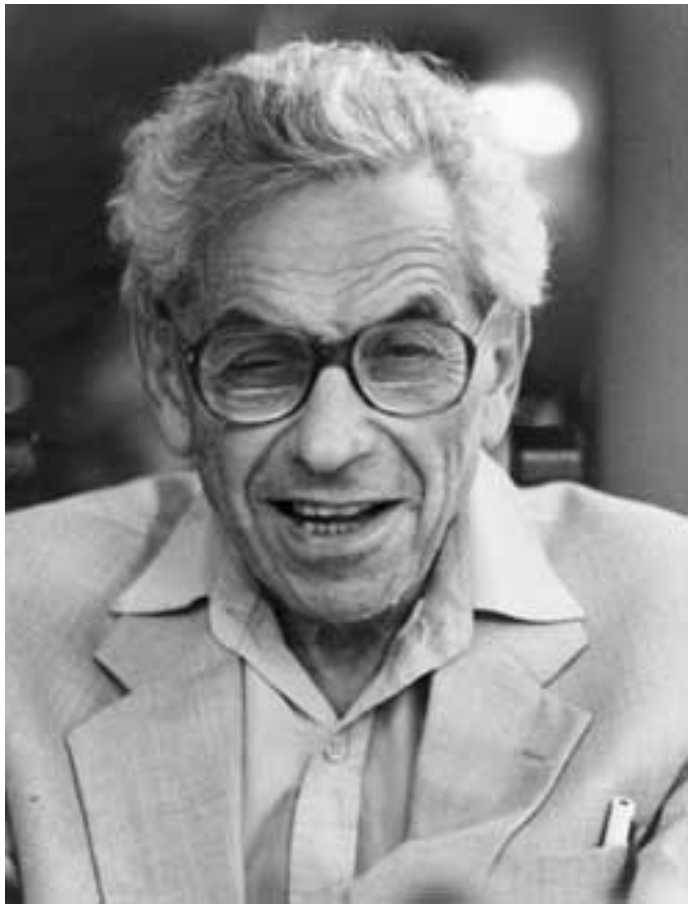
Conclusions of Part II

Languages-theoretical algorithms can be used in order to compute the evaluation at n of several nontrivial arithmetical functions (including characteristic functions) just from the information provided by $\delta(n)$.

Part III

Arithmetic theorems having language-theoretic proofs

"God has the Big Book, the beautiful proofs of mathematical theorems are listed here"



Paul Erdös

"Don't come to me with your pretty proofs. We don't bother with that baby stuff around here !"

Solomon Lefschetz

Arithmetical theorems proved by Language Theory

Theorem (J. M. R. C., 2017)

For all integers $n \geq 2$, if n is not a power of 2 and n is odd-trapezoidal, then $2n$ is the perimeter of a Pythagorean triangle.

Arithmetical theorems proved by Language Theory

Theorem (J. M. R. C., 2017)

For all integers $n \geq 1$, if $2n$ is the perimeter of a Pythagorean triangle, then n has at least two different prime divisors.

Arithmetical theorems proved by Language Theory

Theorem (J. M. R. C., 2017)

For all integers $n \geq 2$, if n is 2-densely divisible and $2n$ is not the perimeter of a Pythagorean triangle then n is a power of 2.

Arithmetical theorems proved by Language Theory

The following result is Theorem 3 in Hartmut F. W. Höft, *On the Symmetric Spectrum of Odd Divisors of a Number*, preprint on-line available at `https://oeis.org/A241561/a241561.pdf`

Theorem (Höft, 2015)

For all $n \geq 1$, there exists at least a middle divisor of n if and only if the number of blocks of n is odd.

Conclusions of Part III

Using language-theoretical relationships, **nontrivial** elementary number-theoretical results can be derived via $\delta(n)$.

An open question

A formal language is *decidable* if there exists a total Turing machine[6] that, when given a finite sequence of symbols as input, accepts it if it belongs to the language and rejects it otherwise.

Is the language $\delta\left(\mathbb{Z}_{\geq 1}\right)$ decidable ?

6. A total Turing machine is a Turing machine that halts for every given input.

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The End

Seshat

goddess of mathematics

Merci !

Thank you !